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Stability Analysis of Distance-Angle Leader-Follower Formation Control*

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***Abstract*— Necessary and sufficient conditions are described for stable distance-angle leader-follower formation control of first- and second-order holonomic and non-holonomic mobile robots. The distance-angle leader-follower formation is a problem of maintaining the desired relative distance and orientation of robots in a group. Our analysis shows that the input constraints on the leader are necessary for stable formation control. These constraints are summarized as follows: 1) In a team of first (second) order holonomic mobile robots, the leader has to be controlled as a first (second) order non-holonomic mobile robot; 2) In a team of first (second) order non-holonomic mobile robots, the control input of the leader must be limited so that the curvature is first (second) order differentiable. We further show that these constraints are sufficient for the followers to maintain formation. Moreover, we present globally asymptotically stable controllers and describe simulation experiments that demonstrate the effectiveness of these controllers.**

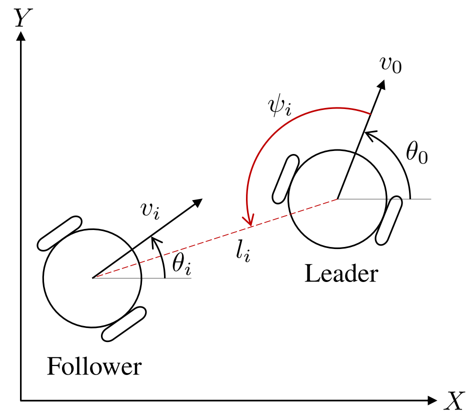

Fig. 1: Distance-angle leader-follower formation of two robots. l_i and ψ_i are the desired relative distance and orientation, respectively.

I. Introduction

Formations of autonomous mobile robots can be used in diverse applications, such as terrain inspection, disaster monitoring, environmental surveillance, and search and rescue. In particular, the control of such formations has been widely studied in relation to unmanned ground vehicles (UGVs) [1], [2], unmanned aerial vehicles (UAVs) [3], [4], [5], autonomous surface vehicles (ASVs) [6], [7], and autonomous underwater vehicles (AUVs) [8], [9].

In formation control, especially in the distance-angle leader-follower approach, a robot that has been assigned as the leader moves toward its goal, while the other robots–the followers–maintain the desired relative distance and orientation to the leader [10], as shown in Figure 1. Various methods of distance-angle leader-follower formation control have been developed, including robust control [11], [12], model predictive control [13], feedback linearization control [5], sliding mode control (SMC) [10], [14], [15], [16], and backstepping control [17], [18], [19]. In particular, backstepping and SMC controllers have been extensively used because they guarantee the global asymptotic stability of the formation.

The formation problem of an uncertain multi-robot system has been addressed with SMC controllers [14], [15]. Here, the controller and observer are integrated into the control scheme to allow formation maneuvers despite uncertainties [14], and a fuzzy compensator is used to approximate the uncertainties [15]. To avoid inconsistencies between different robots' postures, an augmented distance-angle formation was introduced in [16]. However, these studies are for a desired formation that is different from what we consider (they focus on the relative position between the leader's center and the follower's front castor, not the follower's center), and the stability analyses of these two formations are significantly different.

The stability of formations under a backstepping control was proven in [17]. To address the impractical velocity jumps problem inherent to the backstepping control, a bioinspired neurodynamics approach was introduced in [18]. Moreover, a decentralized formation control via a bioinspired neurodynamics approach was proposed in [19]. The proof presented in [17] assumes that the angular velocity of the leader is differentiable. However, it was not discussed whether this assumption is necessary for stability.

In this paper, we present necessary and sufficient conditions for stability. We focus on the control of the leader, which is in contrast to previous studies that focused on the control of the followers. We show that the input constraints on the leader are necessary for stable formation control. These constraints are summarized as follows: 1) In a team of first (second) order holonomic mobile robots, the leader has to be controlled as a first (second) order non-holonomic mobile robot; 2) In a team of first (second) order non-holonomic mobile robots, the control input of the leader must be limited so that the curvature is first (second) order differentiable. We further show that these constraints are sufficient for the followers to maintain formation. Moreover, we developed globally asymptotically stable controllers and

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performed simulation experiments that demonstrate the effectiveness of these controllers.

II. Problem Definition

A. Mobile Robot Dynamics

This paper considers first- and second-order holonomic and non-holonomic mobile robots. The robot dynamics are described as follows:

- First-order holonomic model

$$\dot{p} = v_u \tag{1}$$

- Second-order holonomic model

$$\begin{bmatrix} \dot{p} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ \alpha_u \end{bmatrix} \tag{2}$$

- First-order non-holonomic model

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} v_u^L \cos\theta \\ v_u^L \sin\theta \\ w_u \end{bmatrix} \tag{3}$$

- Second-order non-holonomic model

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{v}^L \\ \dot{w} \end{bmatrix} = \begin{bmatrix} v^L \cos\theta \\ v^L \sin\theta \\ w \\ \alpha_u^L \\ \beta_u \end{bmatrix} \tag{4}$$

In the holonomic models, (1) and (2), p, v, $\alpha \in \mathbf{R}^2$ are the position, velocity, and acceleration of the robot, respectively. In the non-holonomic models, (3) and (4), $(x, y) \in \mathbf{R}^2$, θ, v^L, $w, \alpha^L, \beta \in \mathbf{R}$ are the position, orientation, linear velocity, angular velocity, linear acceleration, angular acceleration of the robot, respectively. The subscript u indicates that the variable is a control input; that is, $v_u, \alpha_u, v_u^L, w_u, \alpha_u^L, \beta_u$ are the control inputs in each model. Note that $p = (x, y)$ and $v = v^L(\cos\theta, \sin\theta)$ hold. For sake of simplicity, the state of a robot corresponding to each model is denoted by q. In other words, $q = p$, $q = [p^\top, v^\top]^\top$, $q = [x, y, \theta]^\top$, and $q = [x, y, \theta, v^L, w]^\top$ correspond to (1), (2), (3), and (4), respectively.

In the non-holonomic models, we assume that forward movement in one direction can translate into backward movement in the opposite direction as follows.

Assumption 1. In non-holonomic models (3) and (4), (v^L, θ) can be converted to $(-v^L, \theta \pm \pi)$.

B. Distance-Angle Leader-Follower Formation Control

We consider a team of mobile robots containing a leader labeled 0 and N followers labeled $1, \ldots, N$. $\mathscr{F}$ denotes the set of followers. The subscript 0 and $i \in \mathscr{F}$ indicate that the variables are the leader's state and follower i's state, respectively. In the distance-angle leader-follower formation, the desired position of each follower is fixed with respect to the leader reference frame. Let $l_i > 0, \psi_i \in [0, 2\pi)$ be the desired relative distance and angle for each follower $i \in \mathscr{F}$. As shown in Figure 1, the desired position of follower i is given by

$$p_i^d = p_0 + l_i R(\psi_i) \frac{v_0}{||v_0||}, \tag{5}$$

where $R(\psi_i)$ is a rotation matrix:

$$R(\psi_i) = \begin{bmatrix} \cos\psi_i & -\sin\psi_i \\ \sin\psi_i & \cos\psi_i \end{bmatrix}.$$

Note that $v_0/\,||v_0|| = (\cos\theta_0, \sin\theta_0)$ holds for the non-holonomic models, (3) and (4). To simplify the discussion in the following section, we assume that the leader has the following constraint.

Assumption 2. The velocity of the leader has a minimum value $v_0^{\min} > 0$; that is, $||v_0|| \geq v_0^{\min}$ and $v^L \geq v_0^{\min}$ hold.

To discuss the stability of the formation, we introduce the desired velocity v_i^d and orientation θ_i^d such that $v_i^d = \dot{p}_i^d$ and $(\cos\theta_i^d, \sin\theta_i^d) = v_i^d/\,\left|\left|v_i^d\right|\right|$. Let q^d be a desired state corresponding to each model. The stability of the formation is defined as the stability of the system $\dot{q}_i^d - \dot{q}_i$.

Definition 1. A formation of a leader and followers is globally asymptotically stable if the following three conditions hold for each $i \in \mathscr{F}$.

1) $q_i^d(t) - q_i(t) = 0 \Rightarrow \dot{q}_i^d(t) - \dot{q}_i(t) = 0$
2) $\forall\epsilon > 0$, $\exists\delta > 0$, $\forall t \geq 0$,
$$\left|\left|q_i^d(0) - q_i(0)\right|\right| < \delta \Rightarrow \left|\left|q_i^d(t) - q_i(t)\right|\right| < \epsilon.$$
3) $\lim_{t\to\infty} \left|\left|q_i^d(t) - q_i(t)\right|\right| = 0$

Finally, we assume that the robots in the team communicate and share information with each other, as follows.

Assumption 3. The leader always sends its (virtual) state and control input to the followers so that each follower knows this information. However, the leader does not monitor the followers; that is, the leader does not know the states of the followers.

Note that, in the following section, we describe that the leader may be controlled as a virtual model that is different from its original one. Therefore, messages sent by the leader may include the virtual state and control input corresponding to the virtual model.

III. Necessary and Sufficient Conditions for Stable Formation Control

This section presents an analysis of the necessary and sufficient conditions for stable formation control. Fig. 2 is an example to illustrate why the input constraint on the leader is necessary for stable formation control. In the figure, there are two first-order non-holonomic mobile robots (3). The blue real line is the trajectory of the leader, and the red dashed line is the trajectory of the desired follower position p_i^d with $l_i = 3, \psi_i = 6\pi/5$. Intuitively, the trajectory of p_i^d is not smooth, so follower i cannot track p_i^d. Hence, the input constraint on the leader is necessary for stable formation control.

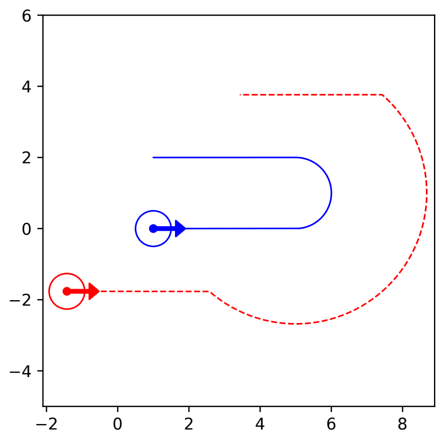


Fig. 2: Example of trajectory of the desired formation position p_i^d with $l_i = 3$ and $\psi_i = \frac{6}{5}\pi$.

If $q_i^d(t)$ is not differentiable, then obviously $\dot{q}_i^d(t) - \dot{q}_i(t) = 0$ does not hold. Thus, the condition that $q_i^d(t)$ is differentiable is necessary for stable formation control, from Definition 1. The differentiability of q_i^d is also closely related to a sufficient condition. If q_i^d is differentiable, stable feedback control laws are available for the system $\dot{q}_i^d - \dot{q}_i$. The following lemma is a necessary and sufficient condition for a stable formation.

Lemma 1. There are control laws for the followers such that the formation is globally asymptotically stable if and only if $q_i^d(t)$ is differentiable for any $t \geq 0$ and $i \in \mathscr{F}$.

The following controller in each model is one of the global asymptotic stable feedback control laws, where q^d is a desired state and where $k_1, k_2, k_3, k_4, k_5 > 0$ are the control gains in each controller.

- Controller of first-order holonomic model (1)

$$v_u = v_u^d + k_1(p^d - p) \tag{6}$$

- Controller of second-order holonomic model (2)

$$\alpha_u = \alpha_u^d + k_1(p^d - p) + k_2(v^d - v) \tag{7}$$

- Controller of first-order non-holonomic model (3)

$$\begin{aligned} v_u^L &= v_u^{Ld} \cos\theta_e + k_1 x_e \\ w_u &= w_u^d + k_2 v_u^{Ld} y_e + k_3 \sin\theta_e \end{aligned} \tag{8}$$

- Controller of second-order non-holonomic model (4)

$$\begin{aligned} \alpha_u^L &= \alpha_u^{Ld} \cos\theta_e - v^{Ld} w_e \sin\theta_e + k_1 x_e \\ &\quad + k_2(v^{Ld} \cos\theta_e - v^L) \\ \beta_u &= \beta_u^d + k_3 \left(\alpha_u^{Ld} y_e + (v^{Ld})^2 \sin\theta_e - v^{Ld} w x_e\right) \\ &\quad + k_4 \sin\theta_e + k_5(k_3 v^{Ld} y_e + w_e)\,, \end{aligned} \tag{9}$$

where $x_e = (x^d - x)\cos\theta + (y^d - y)\sin\theta$, $y_e = -(x^d - x)\sin\theta + (y^d - y)\cos\theta$, $\theta_e = \theta^d - \theta$, and $w_e = w^d - w$.

The proof of the stability of these controllers is shown in Appendix A.

A. First-Order Holonomic Mobile Robots

The following lemma follows from Lemma 1.

Lemma 2. In a team of first-order holonomic mobile robots (1), there are control laws for the followers such that the formation is globally asymptotically stable if and only if $v_{u0}(t)/\,||v_{u0}(t)||$ is differentiable.

Proof. From Lemma 1, we only need to prove that $p_i^d(t)$ is differentiable if and only if $v_{u0}(t)/\,||v_{u0}(t)||$ is differentiable. p_0 is differentiable ($\dot{p}_0 = v_{u0}$). Thus, in Equation (5), $p_i^d(t)$ is differentiable if and only if $v_{u0}(t)/\,||v_{u0}(t)||$ is differentiable. □

Consider a first-order holonomic mobile robot in which $v_u(t)/\,||v_u(t)||$ is differentiable. This model can be represented by the following equation.

$$\begin{bmatrix} \dot{p} \\ \frac{d}{dt}\frac{v_u}{||v_u||} \end{bmatrix} = \begin{bmatrix} v_u \\ c_u \frac{v_u^\perp}{||v_u||} \end{bmatrix}, \tag{10}$$

where $c_u \in \mathbf{R}$ and $v_u^\perp = R(\pi/2)v_u$. The derivation of Equation (10) is shown in Appendix B.

Consider the following model, which is a variant of the first-order non-holonomic model, (3).

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \frac{d}{dt}\cos\theta \\ \frac{d}{dt}\sin\theta \end{bmatrix} = \begin{bmatrix} v_u^L \cos\theta \\ v_u^L \sin\theta \\ -w_u \sin\theta \\ w_u \cos\theta \end{bmatrix} \tag{11}$$

Obviously, when $v_u = v_u^L(\cos\theta, \sin\theta)$ and $c_u = w_u$, model (10) is equivalent to the variant (11), so model (10) is equivalent to the first-order non-holonomic model (3). Hence, Lemma 2 can be rewritten as follows.

Theorem 1. In a team of first-order holonomic mobile robots (1), there are control laws for the followers such that the formation is globally asymptotically stable if and only if the leader is controlled as a first-order non-holonomic mobile robot (3).

Finally, we show the velocity of the desired position p_i^d when the leader is controlled as a first-order non-holonomic model, (3).

$$v_i^d = v_{u0}^L \begin{bmatrix} \cos\theta_0 \\ \sin\theta_0 \end{bmatrix} + w_{u0} l_i R_i(\psi_i) \begin{bmatrix} -\sin\theta_0 \\ \cos\theta_0 \end{bmatrix} \tag{12}$$

We can obtain a globally asymptotically stable formation law by substituting the desired position (5) and velocity (12) into the feedback controller (6).

B. Second-Order Holonomic Mobile Robots

We will follow the same procedure as in Section III-A to derive a necessary and sufficient condition for stable formation control of second-order holonomic mobile robots. The following lemma holds from Lemma 1.

Lemma 3. Consider $a_u, b_u \in \mathbf{R}$ such that

$$\alpha_u = a_u \frac{v}{||v||} + b_u \frac{v^\perp}{||v||}. \tag{13}$$

In a team of second-order holonomic mobile robots (2), there are control laws for the followers such that the formation is globally asymptotically stable if and only if $b_{u0}(t)$ is differentiable.

Proof. Obviously, $p_i^d(t)$ is differentiable. Thus, from Lemma 1, we only need to prove that $v_i^d(t)$ is differentiable if and only if $b_{u0}(t)$ is differentiable.

The following equation holds because $v_0 \cdot v_0^{\perp} = 0$.

$$\begin{aligned} v_i^d &= v_0 + l_i R(\psi_i) \frac{1}{||v_0||^3} \left(\alpha_{u0} ||v_0||^2 - v_0 (v_0 \cdot \alpha_{u0}) \right) \\ &= v_0 + b_{u0} l_i R(\psi_i) \frac{v_0^{\perp}}{||v_0||^2} \end{aligned}$$

v_0 and $\frac{v_0^{\perp}}{||v_0||^2}$ are differentiable. Hence, $v_i^d(t)$ is differentiable if and only if $b_{u0}(t)$ is differentiable. □

The second-order holonomic model in which $b_u(t)$ is differentiable can be represented by the following equation.

$$\begin{bmatrix} \dot{p} \\ \dot{v} \\ \dot{b}_u \end{bmatrix} = \begin{bmatrix} v \\ a_u \frac{v}{||v||} + b_u \frac{v^{\perp}}{||v||} \\ c_u \end{bmatrix}, \tag{14}$$

where $c_u \in \mathbf{R}$.

Consider the following model, which is a variant of the second-order non-holonomic model, (4).

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \frac{d}{dt} v^L \cos\theta \\ \frac{d}{dt} v^L \sin\theta \\ \dot{w} \end{bmatrix} = \begin{bmatrix} v^L \cos\theta \\ v^L \sin\theta \\ \alpha_u^L \cos\theta - w v^L \sin\theta \\ \alpha_u^L \sin\theta + w v^L \cos\theta \\ \beta_u \end{bmatrix} \tag{15}$$

When $v = v^L(\cos\theta, \sin\theta)$, $a_u = \alpha_u^L$, $b_u = w v^L$, and $c_u = \beta_u v^L + w \alpha_u^L$ hold, model (14) is equivalent to variant model (15), so model (14) is equivalent to second-order non-holonomic model (4). Hence, Lemma 3 can be rewritten as follows.

Theorem 2. In a team of second-order holonomic mobile robots (2), there are control laws for the followers such that the formation is globally asymptotically stable if and only if the leader is controlled as a second-order non-holonomic mobile robot (4).

Finally, we show the acceleration of the desired position p_i^d when the leader is controlled as a second-order non-holonomic model, (4).

$$\alpha_i^d = f_i(q_0, \alpha_{u0}^L) \begin{bmatrix} \cos\theta_0 \\ \sin\theta_0 \end{bmatrix} + g_i(q_0, \beta_{u0}) \begin{bmatrix} -\sin\theta_0 \\ \cos\theta_0 \end{bmatrix}, \tag{16}$$

where

$$\begin{aligned} f_i(q_0, \alpha_{u0}^L) &= \alpha_{u0}^L - w_0^2 l_i R(\psi_i) \\ g_i(q_0, \beta_{u0}) &= v_0^L w_0 + \beta_{u0} l_i R(\psi_i). \end{aligned}$$

We can obtain a globally asymptotically stable formation control law by substituting the desired position (5), velocity (12), and acceleration (16) into the feedback controller (7).

C. First-Order Non-Holonomic Mobile Robots

First, let us derive the linear velocity and orientation of the desired point $p_i^d(t)$ in order to discuss the stability of the formation. The desired linear velocity v_i^{Ld} equals $||\dot{p}_i^d||$, where $\dot{x}_i^d$ and $\dot{y}_i^d$ are

$$\begin{aligned} \dot{x}_i^d &= \left(v_{u0}^L - w_{u0} l_i \sin\psi_i\right) \cos\theta_0 - w_{u0} l_i \cos\psi_i \sin\theta_0 \\ \dot{y}_i^d &= \left(v_{u0}^L - w_{u0} l_i \sin\psi_i\right) \sin\theta_0 + w_{u0} l_i \cos\psi_i \cos\theta_0. \end{aligned} \tag{17}$$

Hence, the velocity is given by

$$v_i^{Ld} = \sqrt{\left(v_{u0}^L - w_{u0} l_i \sin\psi_i\right)^2 + \left(w_{u0} l_i \cos\psi_i\right)^2}. \tag{18}$$

Note that $v_i^{Ld} > 0$ holds if $v_{u0}^L > 0$ and $\psi_i \neq \frac{\pi}{2}, \frac{3\pi}{2}$ hold. Let

$$\varphi_i = s(v_{u0}^L - w_{u0} l_i \sin\psi_i) \tan^{-1} \left(\frac{w_{u0} l_i \cos\psi_i}{\left| v_{u0}^L - w_{u0} l_i \sin\psi_i \right|} \right), \tag{19}$$

where s is the sign function. $\varphi_i \in [-\pi, \pi]$ satisfies

$$\begin{aligned} \cos\varphi_i &= (v_{u0}^L - w_{u0} l_i \sin\psi_i) / v_i^{Ld} \\ \sin\varphi_i &= w_{u0} l_i \cos\psi_i / v_i^{Ld}. \end{aligned}$$

Thus, the dynamics (17) can be rewritten from the sum and difference identities of angles as follows.

$$\begin{aligned} \dot{x}_i^d &= v_i^{Ld} \cos(\theta_0 + \varphi_i) \\ \dot{y}_i^d &= v_i^{Ld} \sin(\theta_0 + \varphi_i) \end{aligned}$$

Hence, the desired orientation is given by

$$\theta_i^d = \theta_0 + \varphi_i. \tag{20}$$

Note that if $\psi_i = \frac{\pi}{2}, \frac{3\pi}{2}$ holds, then $\theta_i^d = \theta_0$ holds. This means that $\psi_i = \frac{\pi}{2}, \frac{3\pi}{2}$ is a sufficient condition for stable formation control. Thus, we will assume $\psi_i \neq \frac{\pi}{2}, \frac{3\pi}{2}$ in the following discussion. The following lemma follows from Lemma 1.

Lemma 4. In a team of first-order non-holonomic mobile robots (3), there are control laws for the followers such that the formation is globally asymptotically stable if and only if the curvature of the leader trajectory is differentiable; that is, w_{u0}/v_{u0}^L is differentiable.

Proof. From Lemma 1, we only need to prove that θ_i^d is differentiable if and only if w_{u0}/v_{u0}^L is differentiable.

Let

$$Q(t) = \frac{w_{u0}(t) l_i \cos\psi_i}{v_{u0}^L(t) - w_{u0}(t) l_i \sin\psi_i}.$$

The following conditions hold: 1) θ_i^d is differentiable if and only if φ_i is differentiable; 2) φ_i is differentiable if and only if Q is differentiable.

Consider the differentiability of Q. To simplify the notation, let a and b be $l_i \cos\psi_i$ and $l_i \sin\psi_i$, respectively. The following equation is derived from $Q(t + \Delta t) - Q(t)$.

$$\frac{a v_{u0}^L(t+\Delta t) v_{u0}^L(t) \left(\frac{w_{u0}(t+\Delta t)}{v_{u0}^L(t+\Delta t)} - \frac{w_{u0}(t)}{v_{u0}^L(t)} \right)}{\left(v_{u0}^L(t+\Delta t) - b w_{u0}(t+\Delta t)\right) \left(v_{u0}^L(t) - b w_{u0}(t)\right)}$$

Hence, θ_i^d is differentiable if and only if w_{u0}/v_{u0}^L is differentiable. $\square$

To handle the curvature constraint easily, we consider a first-order non-holonomic model in which the curvature is differentiable, as follows.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\kappa} \end{bmatrix} = \begin{bmatrix} v_u^L \cos\theta \\ v_u^L \sin\theta \\ v_u^L \kappa \\ \gamma_u \end{bmatrix}, \tag{21}$$

where $\kappa = w/v_u^L$ is the curvature. Lemma 4 can be rewritten as follows.

Theorem 3. In a team of first-order non-holonomic mobile robots (3), there are control laws for the followers such that the formation is globally asymptotically stable if and only if the leader is controlled as a first-order non-holonomic mobile robot with differentiable curvature (21).

Finally, we show the angular velocity of the desired position p_i^d when the leader is controlled in accordance with the model (21).

$$w_i^d = v_{u0}^L \kappa_0 + \gamma_{u0} \left(\frac{v_{u0}^L}{v_i^{Ld}} \right)^2 l_i \cos\psi_i \tag{22}$$

We can obtain a globally asymptotically stable formation control law by substituting the desired position (5), linear velocity (18), orientation (20), and angular velocity (22) into the feedback controller (8)

D. Second-Order Non-Holonomic Mobile Robots

The following lemma follows from Lemma 1.

Lemma 5. In a team of second-order non-holonomic mobile robots (4), there are control laws for the followers such that the formation is globally asymptotically stable if and only if the curvature of the leader trajectory is second-order differentiable.

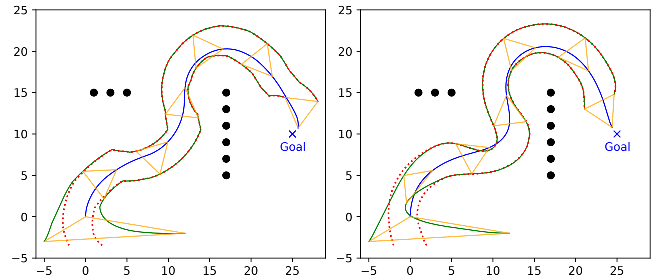
Proof. We only need to prove $w_i^d(t)$ is differentiable if and only if curvature $\kappa_0(t) = w_0(t)/v_0^L(t)$ is second-order differentiable. Because $v_0^L, \kappa_0, v_i^{Ld}$ is differentiable in (22), w_i^d is differentiable if and only if γ_0 is differentiable. $\square$

To handle the curvature constraint easily, we consider a second-order non-holonomic model in which the curvature is second-order differentiable, as follows.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{v} \\ \dot{\kappa} \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} v^L \cos\theta \\ v^L \sin\theta \\ v^L \kappa \\ \alpha_u^L \\ \gamma \\ \xi_u \end{bmatrix} \tag{23}$$

Lemma 4 can be rewritten as follows.

Theorem 4. In a team of second-order non-holonomic mobile robots (4), there are control laws for the followers such that the formation is globally asymptotically stable if and only if the leader is controlled as a second-order non-holonomic mobile robot with second-order differentiable curvature (23).



(a) First-order holonomic model. (b) Second-order holonomic model.

Fig. 3: Trajectories of three holonomic mobile robots with our formation controllers.

Finally, we show the linear and angular accelerations of the desired position p_i^d when the leader is controlled in accordance with the model (23).

$$\begin{aligned} \alpha_i^{Ld} &= \alpha_{u0}^L s_{vi} + \gamma_0 \frac{v_0^L}{s_{vi}} \left(\kappa_0 l_i^2 - l_i \sin\psi_i \right) \\ \beta_i^d &= \alpha_{u0}^L \kappa_0 + v_0^L \gamma_0 \\ &\quad + \left\{ \frac{\xi_{u0}}{s_{vi}^2} + \frac{2\gamma_0^2}{s_{vi}^4} \left(\kappa_0 l_i^2 - l_i \sin\psi_i \right) \right\} l_i \cos\psi_i, \end{aligned} \tag{24}$$

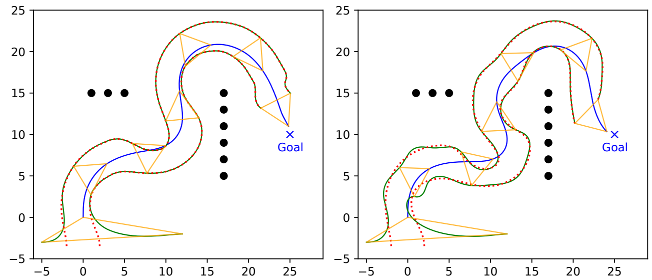
where $s_{vi} = v_i^{Ld}/v_0^L$. We can obtain a globally asymptotically stable formation control law by substituting the desired position (5), linear velocity (18), orientation (20), angular velocity (22), and linear and angular accelerations (24) into the feedback controller (9).

IV. Experimental Results

We demonstrated our controllers in numerical simulations. The problem settings are as follows.

- Two followers track the leader while keeping a triangle formation. Let $l_1 = l_2 = 4$, $\psi_1 = \frac{2}{3}\pi$ and $\psi_2 = \frac{4}{3}\pi$.
- All control gains of the followers' feedback controllers are 1; that is, $k_1 = k_2 = k_3 = k_4 = k_5 = 1$ in each feedback controller (6), (7), (8), (9).
- The leader moves toward its destination while avoiding collisions with obstacles. The navigation algorithm of the leader is the dynamic window approach (DWA), which is a well-known collision avoidance approach for non-holonomic mobile robots [20].
- In DWA, the leader is considered to have a virtual body with a radius of $\max_{i \in \mathscr{F}} l_i$ as a means for the followers to avoid colliding with obstacles.

Fig. 3a shows the trajectories of first-order holonomic mobile robots (1) with our controllers. The blue and green lines are trajectories of the leader and the followers, respectively. The red dotted line is a trajectory of each desired follower position q_i^d. The orange triangle is the formation of the robots at each time. The leader that was controlled in accordance with the first-order non-holonomic model (3) based on DWA



(a) First-order non-holonomic model. (b) Second-order non-holonomic model.

Fig. 4: Trajectories of three non-holonomic mobile robots with our formation controllers.

reached its goal while avoiding collisions between the three robots and the obstacles represented by black circles. The followers with the feedback controller (6) tracked the leader while keeping the triangle formation.

Figures 3b, 4a, and 4b show the trajectories of second-order holonomic robots (2) and the trajectories of first- and second-order non-holonomic mobile robots (3) and (4), respectively. These figures show that our controllers for each model were asymptotically stable.

V. Conclusion

We described necessary and sufficient conditions for a stable distance-angle leader-follower formation with first- and second-order holonomic and non-holonomic mobile robots. We also presented globally asymptotically stable controllers. The simulation results for a team of three robots in each model showed that a triangle formation could be achieved, demonstrating the effectiveness of our controllers.

Appendix

A. Proof of Stability of Feedback Controllers

We prove the stability of the feedback controllers using LaSalle's invariance principle. Consider the following Lyapunov function candidates $V(q) \geq 0$ for the feedback controllers (6), (7), (8), and (9), respectively.

$$V_1(q) = \frac{1}{2}\left|\left|p^d - p\right|\right|^2$$

$$V_2(q) = \frac{k_1}{2}\left|\left|p^d - p\right|\right|^2 + \frac{1}{2}\left|\left|v^d - v\right|\right|^2$$

$$V_3(q) = \frac{k_2}{2}\left(x_e^2 + y_e^2\right) + (1 - \cos\theta_e)$$

$$V_4(q) = \frac{k_3}{2}\left(x_e^2 + y_e^2\right) + (1 - \cos\theta_e) + \frac{k_3}{2k_1}(v^{Ld}\cos\theta_e - v^L)^2 + \frac{1}{2k_4}(k_3 v^{Ld} y_e + w_e)^2$$

In each function, $q = q^q$ if and only if $V(q) = 0$.

$\dot{V}_1(q) = -k_1\left|\left|p^d - p\right|\right|^2 \leq 0$ holds for the first-order holonomic model. Obviously, the feedback controller (6) is globally asymptotically stable.

In the second-order holonomic model, $\dot{V}_2(q) = -k_2\left|\left|v^d - v\right|\right|^2 \leq 0$ holds. Any subset of the set $\{q \,|\, \dot{V}_2(q) = 0 \wedge V_2(q) \neq 0\}$ which is simply the set $\{q \,|\, v = v^d \wedge p \neq p^d\}$ is not positively invariant because $\alpha \neq \alpha^d$ if $v = v^d$ and $p \neq p^d$. Hence, the feedback controller (7) is globally asymptotically stable.

Suppose that $v_u^{Ld} > 0$. The following equations hold for the first-order non-holonomic model.

$$\dot{x}_e = v_u^{Ld}\cos\theta_e + w_u y_e - v_u^L \quad (25)$$

$$\dot{y}_e = v_u^{Ld}\sin\theta_e - w_u x_e \quad (26)$$

Therefore, $\dot{V}_3(q) = -k_1 k_2 x_e^2 - k_3\sin^2\theta_e \leq 0$, and $\dot{V}_3(q) = 0$ if and only if $q \in \{q \,|\, x_e = \theta_e = 0\}$. The maximal positively invariant set only contains $q \neq q^d$ because $w_u^d \neq w_u$ if $x_e = \theta_e = 0$ and $y_e \neq 0$. Hence, the feedback controller (8) is globally asymptotically stable.

In the second-order non-holonomic model, $\dot{V}_4(q) = -\frac{k_2 k_3}{k_1}(v^{Ld}\cos\theta_e - v^L)^2 - \frac{k_5}{k_4}(k_3 v^{Ld} y_e + w_e)^2 \leq 0$ holds. If q is an element of the positively invariant set, then q satisfies the following conditions.

$$v^{Ld}\cos\theta_e - v^L = \frac{d}{dt}\left(v^{Ld}\cos\theta_e - v^L\right) = 0 \quad (27)$$

$$k_3 v^{Ld} y_e + w_e = \frac{d}{dt}\left(k_3 v^{Ld} y_e + w_e\right) = 0 \quad (28)$$

$x_e = 0$ is derived from equations (9) and (27). Similarly, $\theta_e = 0$ is derived from equations (9) and (28). Moreover, $v^L = v^{Ld}$ holds because $v^{Ld}\cos\theta_e - v^L = 0$ and $\theta_e = 0$. Thus, if q is an element of the positively invariant set, then q also satisfies the conditions $\dot{\theta}_e = w_e = 0$. If $w_e = 0$, then $y_e = 0$ holds, because $k_3 v^{Ld} y_e + w_e = 0$. Hence, the feedback controller (9) is globally asymptotically stable because the maximum positively invariant set only contains $q = q^d$. □

B. Derivation of Model (10)

Consider $a(t), b(t) \in \mathbf{R}$ such that it satisfies $v(t+\Delta t) = a(t+\Delta t)\frac{v(t)}{||v(t)||} + b(t+\Delta t)\frac{v^\perp(t)}{||v(t)||}$, where the subscript u is omitted to simplify the notation. The following equation holds.

$$\frac{v(t+\Delta t)}{||v(t+\Delta t)||} - \frac{v(t)}{||v(t)||} = \frac{\left(a - \sqrt{a^2+b^2}\right)v(t) + b v^\perp(t)}{||v(t)||\sqrt{a^2+b^2}},$$

where $a = a(t+\Delta t), b = b(t+\Delta t)$. Obviously, $c(t)$ exists such that it satisfies

$$\lim_{\Delta t \to 0}\frac{b(t+\Delta t)}{\Delta t} = c(t)$$

because $\frac{v(t)}{||v(t)||}$ is differentiable. Moreover, $a - \sqrt{a^2+b^2} = -b^2/(a + \sqrt{a^2+b^2})$ holds, because $(a + \sqrt{a^2+b^2})(a - \sqrt{a^2+b^2}) = -b^2$. Moreover, $\lim_{\Delta t \to 0}\frac{b^2(t+\Delta t)}{\Delta t} = 0$, so the following equation holds.

$$\frac{d}{dt}\frac{v(t)}{||v(t)||} = \lim_{\Delta t \to 0}\frac{b(t+\Delta t)}{\Delta t}\frac{v^\perp(t)}{||v(t)||} = c(t)\frac{v^\perp(t)}{||v(t)||}$$

□

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