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Parameter Identifying Disturbance Rejection Control with Asymptotic Error Convergence

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Abstract—**In this paper, a new kind of adaptive controller for the problem of output feedback tracking is proposed on the basis of the Active Disturbance Rejection Control (ADRC) paradigm. The controller is synthesized for the systems linear in parameters by combining the classic ADRC algorithm with a recent Parameter Identifying Extended State Observer (PIESO) which employs a gradient adaptation law to actively identify the parameters of the plant. By means of the Lyapunov analysis, the asymptotic convergence of tracking, estimation, and identification errors is proved in the nominal case and the stability conditions of the closed-loop system are formulated. The theoretical analysis is complemented by simulation and experimental results of the proposed controller.**

Index Terms—**Robust/Adaptive Control, Calibration and Identification, Formal Methods in Robotics and Automation**

I. Introduction

The problem of the adaptive output feedback control is one of the essential topics of interest in automatic control. Numerous solutions to the problem of simultaneous control, state estimation, and parameter identification in the presence of a limited measurability of the plant state have been proposed in the literature. Since a modelling error typically cannot be determined directly, some substitute for it is commonly employed in adaptation schemes. Known methods include adaptive control laws synthesized on the basis of the tracking error by means of the Lyapunov approach [1], [2] or multiple variants of the classic Model Reference Adaptive Control (MRAC) algorithm with adaptation based on either tracking [3], [4] or state estimation error [5]. Other class of the algorithms employs the state filters to support the adaptation procedure [6], [7].

The alternative approach based on the Active Disturbance Rejection Control (ADRC) paradigm [8]–[10] is embraced in this paper. The application oriented research in the field of ADRC is conducted mainly in two major directions – to weaken some of the fundamental assumptions concerning the required knowledge of the plant [11]–[13] or to improve the performance of the method in the presence of certain predefined types of disturbances affecting the system. Several adaptive controllers that work in such a regime have been reported to address the latter of these problems. In the works [14], [15] standard Extended State Observer (ESO) was employed and the model identification error was algebraically derived from the total disturbance estimate to be later used in the gradient identification law. Adaptive ADRC algorithms with adaptation laws based on the tracking error have also been proposed [16]–[18]. Recently, some new results on adaptive control taking advantage of the ability of ESO to online estimate the modeling error have been presented in [19], [20] and [21]. However, these algorithms are dedicated for specific applications only or do not offer asymptotic convergence of the tracking errors. There are also known adaptive ADRC methods [22], [23] based on the full state feedback, for which asymptotic tracking is proved, while the convergence of parameter estimates is not guaranteed.

In this paper a new solution for the problem of the feedback adaptive control is proposed on the basis of ADRC method in the form of Parameter Identifying Disturbance Rejection Control (PIDRC) algorithm that uses the recent PIESO [24] to adaptively estimate the state of the plant. The considered method combines the ADRC algorithm with the classic gradient adaptation law to online identify the unknown parameters of the plant. Specifically, the observer is designed by incorporating adaptive model of the system taking advantage of the identified parameter estimates, what allows one to interpret the total disturbance estimate as a measure of a modeling error. A gradient adaptation law is formulated on the basis of this estimate and the control law is synthesized employing the identified model of the disturbance. It is shown that the proposed method ensures asymptotic convergence of tracking, estimation and identification errors along with the improved transient performance, combining the merits of the ADRC method and adaptive algorithms. Simultaneously, the proposed adaptive observer is similar in complexity to the standard ESO and the employed disturbance model is of the same kind as used in classic adaptive schemes. Thus, the proposed method does not introduce a significant overhead in comparison with standard ESO and adaptive algorithms from both implementation and mathematical standpoints.

Some advantages of the proposed solution over the classic output feedback control methods may be distinguished. Firstly, the conceptual difference lies in the embracement of the total disturbance estimate as the basis of the adaptation law. Due to the specifically redefined dynamics of the plant, the observer is able to directly estimate the modeling error. As a result, the proposed adaptation law has a straightforward interpretation and is conceptually separated from the tracking

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and state estimation errors, what is not a case in multiple classic algorithms. The underlying philosophy of the method is thus different from standard MRAC-like approaches and has not been previously studied in detail. Secondly, the proposed solution employs the estimated modeling error not only in the adaptation scheme, but also directly in the essential part of control law. The transient performance of the closed-loop system is thus improved in comparison with the classic adaptive methods, in which the reduction of the tracking error is strictly correlated with the process of the parameter identification. The approach suggested in this paper takes advantage of the estimate of the lumped modeling error in the initial transient phase to attenuate the negative impact of this error on the tracking accuracy and employs the adaptive scheme mainly to ensure the asymptotic convergence of the tracking errors. As the adaptation speed is often deemed unsatisfactory in practical applications of the adaptive controllers, the separation of the performance in the transient state from the identification progress enables a significant improvement of the closed-loop system performance. Moreover, in contrast to multiple schemes proposed in the literature, which consider only the problem of tracking errors convergence, the asymptotic convergence of the estimation and identification errors are also established in the proposed framework under standard persistency of excitation conditions. If this condition is fully satisfied, the presented study yields the strict Lyapunov function as a basis of stability analysis.

To the best of authors knowledge, the method proposed in this paper is the first ADRC-based adaptive solution with proven asymptotic convergence of the errors despite initial parameter uncertainty. In addition, the presented method employs a novel and unorthodox approach with underlying philosophy being distinctly different from classic MRAC-like approaches and which has not yet been extensively investigated in the available literature.

The structure of the rest of the paper is as follows. In Section II the main problem of this paper is formulated and the requirements for the applicability of the proposed approach are given. In Section III the main results are presented and a theoretical analysis of the considered algorithm is given. Sections IV and V contain the results of the simulations and experiments are presented as a validation of the considered algorithm. Section VI concludes the paper.

Throughout this paper, for any constant $\kappa \geq 2 \in \mathbb{N}$ denote $\boldsymbol{A}_\kappa = \begin{bmatrix}\boldsymbol{0}_{\kappa\times 1} & \boldsymbol{a}_2\end{bmatrix} \in \mathbb{R}^{\kappa\times\kappa}$ with $\boldsymbol{a}_2 = \begin{bmatrix}\boldsymbol{I}_{\kappa-1} & \boldsymbol{0}_{\kappa-1\times 1}\end{bmatrix}^T$. Let $\boldsymbol{b}_\kappa = \begin{bmatrix}\boldsymbol{0}_{1\times\kappa-1} & 1\end{bmatrix}^T \in \mathbb{R}^\kappa$, $\boldsymbol{c}_\kappa = \begin{bmatrix}1 & \boldsymbol{0}_{1\times\kappa-1}\end{bmatrix}^T \in \mathbb{R}^\kappa$, $\boldsymbol{d}_\kappa = \begin{bmatrix}\boldsymbol{0}_{1\times\kappa-2} & 1 & 0\end{bmatrix}^T \in \mathbb{R}^\kappa$. Moreover, define matrix $\boldsymbol{\Lambda}_\kappa = \begin{bmatrix}\boldsymbol{I}_\kappa & \boldsymbol{0}_{\kappa\times 1}\end{bmatrix} \in \mathbb{R}^{\kappa\times\kappa+1}$. Throughout the paper $\|\cdot\|$ denotes 2-norm.

II. Problem formulation

The problem of trajectory tracking for a class of dynamic systems is considered. Let the nominal dynamic system be expressed as

$$\dot{\boldsymbol{x}} = \boldsymbol{A}_n\boldsymbol{x} + \boldsymbol{b}_n\big(bu + \boldsymbol{\psi}(t,\boldsymbol{x})\boldsymbol{\theta}\big), \quad y = \boldsymbol{c}_n^T\boldsymbol{x}, \tag{1}$$

where $\boldsymbol{x} = \begin{bmatrix}x_1 & \dots & x_n\end{bmatrix}^T \in \mathbb{R}^n$ represents the state of the system which is not available for measurement, $u \in \mathbb{R}$ is a control input, $b \in \mathbb{R}\setminus\{0\}$ is some known constant input gain coefficient, $\boldsymbol{\psi}: \mathbb{R}_{\geq 0}\times\mathbb{R}^n \to \mathbb{R}^{1\times k}$ is called the regressor of the system and $\boldsymbol{\theta} = \begin{bmatrix}\theta_1 & \dots & \theta_k\end{bmatrix}^T \in \mathbb{R}^k$ denotes the unknown parameters. The term $y \in \mathbb{R}$ stands for the measurable output of the plant. Define now a reference trajectory

$$\boldsymbol{x}_d(t) = \begin{bmatrix}x_d(t) & \frac{d}{dt}x_d(t) & \dots & \frac{d^{n-1}}{dt^{n-1}}x_d(t)\end{bmatrix}^T, \tag{2}$$

with $x_d(t) \in \mathbb{R}$ being some known function of time, which is smooth enough. In order to investigate the properties of the proposed solution for different degrees of regressor excitation, the approach proposed in [2] is embraced and combined with the notions of [24] and earlier works on ADRC and adaptive control [25], [26], to formulate the following assumptions

Assumption 1: Let $\boldsymbol{\psi}(t,\boldsymbol{x})$ be Lipschitz with respect to the second argument, thus for any $t \geq 0$ and $\boldsymbol{x}_a, \boldsymbol{x}_b \in \mathbb{R}^n$,

$$\|\boldsymbol{\psi}(t,\boldsymbol{x}_a) - \boldsymbol{\psi}(t,\boldsymbol{x}_b)\| \leq \psi_M\|\boldsymbol{x}_a - \boldsymbol{x}_b\| \tag{3}$$

with $\psi_M \in \mathbb{R}_{\geq 0}$ being some nonnegative constant. Let also $\boldsymbol{x}_d(t)$ be chosen such that $\boldsymbol{\psi}(t,\boldsymbol{x}_d)$ is continuous and at least once differentiable on the reference trajectory, with the regressor and its derivative, evaluated on the reference trajectory, being bounded. Thus, let constant ψ_M also satisfy

$$\max_{t\geq 0}\left(\|\boldsymbol{\psi}(t,\boldsymbol{x}_d)\|, \|\tfrac{d}{dt}\boldsymbol{\psi}(t,\boldsymbol{x}_d)\|\right) \leq \psi_M. \tag{4}$$

Assumption 2: Let $\boldsymbol{x}_d(t)$ be chosen such that there exists nonsingular constant matrix $\boldsymbol{S} \in \mathbb{R}^{k\times k}$ such that

$$\boldsymbol{\psi}(t,\boldsymbol{x}_d)\boldsymbol{S} = \begin{bmatrix}\bar{\boldsymbol{\psi}}(t,\boldsymbol{x}_d) & \boldsymbol{0}_{1\times k-p}\end{bmatrix} \tag{5}$$

with $p \leq k$ being some nonnegative integer and $\bar{\boldsymbol{\psi}}: \mathbb{R}_{\geq 0}\times\mathbb{R}^n \to \mathbb{R}^{1\times p}$ satisfying

$$\int_t^{t+T_{\mathrm{PE}}} \bar{\boldsymbol{\psi}}^T(\tau,\boldsymbol{x}_d(\tau))\bar{\boldsymbol{\psi}}(\tau,\boldsymbol{x}_d(\tau))d\tau \geq \mu\psi_M\boldsymbol{I} \tag{6}$$

for some constants $\mu, T_{\mathrm{PE}} \in \mathbb{R}_+$ and every $t \geq 0$. Without loss of generality assume that $\|\boldsymbol{S}\| = 1$.

Equation (4) corresponds to the boundedness requirement specified with respect to the regressor and its time derivative, while (3) imposes limitation on the rate of change of the nonlinear terms of the regressor. Notably, these bounds may be arbitrarily large resulting only in the requirement for the higher gains of the observer and controller, as will be shown by forthcoming analysis. In addition, the relationship (6) defines a constraint, commonly known as the persistent excitation (PE) condition, imposed on the regressor of the system. Due to the use of $\boldsymbol{S}$ matrix, the forthcoming analysis covers a wide class of systems, including plants fully satisfying the classic PE condition (by taking $\boldsymbol{S} = \boldsymbol{I}$), systems with some redundancy in the regressor design (by taking $\boldsymbol{S} \neq \boldsymbol{I}$), or plants which does not satisfy the PE condition (by setting $\psi_M = 0$) what may correspond e.g. to the problem of adaptive regulation with a constant reference trajectory. It is essential that these assumptions are formulated on the basis of reference trajectory only and no significant requirements on transient states are being imposed. Thus, all of the assumptions presented here can be verified in advance and the presented analysis can be strictly satisfied in the practical scenarios.

Corollary 1 (from [27]–[29]): Define a persistently excited matrix in the form of

$$\boldsymbol{M}(t, \boldsymbol{x}_d) = \int_t^{\infty} e^{t-\tau} \bar{\boldsymbol{\psi}}^T(\tau, \boldsymbol{x}_d(\tau)) \bar{\boldsymbol{\psi}}(\tau, \boldsymbol{x}_d(\tau)) d\tau, \quad (7)$$

with derivative given by

$$\tfrac{d}{dt}\boldsymbol{M}(t, \boldsymbol{x}_d) = \boldsymbol{M}(t, \boldsymbol{x}_d) - \bar{\boldsymbol{\psi}}^T(t, \boldsymbol{x}_d(t)) \bar{\boldsymbol{\psi}}(t, \boldsymbol{x}_d(t)). \quad (8)$$

If Assumptions 1 and 2 hold, then for any vector $\boldsymbol{v} \in \mathbb{R}^p$, matrix $\boldsymbol{M}(t, \boldsymbol{x}_d)$ satisfies

$$\mu\psi_M e^{-T_{\mathrm{PE}}}\|\boldsymbol{v}\|^2 \leq \boldsymbol{v}^T\boldsymbol{M}\boldsymbol{v} \leq \psi_M^2\|\boldsymbol{v}\|^2. \quad (9)$$

III. Parameter Identifying Disturbance Rejection Controller

According to the ADRC paradigm, the standard approach to the problem of trajectory tracking for the system (1) would be to consider entire expression $\boldsymbol{\psi}(t, \boldsymbol{x})\boldsymbol{\theta}$ as an unknown total disturbance to be estimated by the ESO. This estimate would be in turn injected into the control law to roughly compensate for these unknown dynamics and guarantee the convergence of the state $\boldsymbol{x}$ to some neighborhood of the reference trajectory $\boldsymbol{x}_d$. However, if the general structure of $\boldsymbol{\psi}(t, \boldsymbol{x})$ is known, an alternative approach can be proposed to improve the control performance. In particular, the recently introduced PIESO [24] can be employed to simultaneously estimate the state of the system, a total disturbance, and the system parameters. These could be then used to formulate a control law, which depends on the estimate of the lumped total disturbance, while incorporating the disturbance model being continuously improved as adaptation progresses.

Consider the dynamic system given by (1). Following [24], the PIESO for such a system can be designed by introducing an additional state variable $\delta = \boldsymbol{\psi}(t, \boldsymbol{x}_d)\big(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\big) \in \mathbb{R}$ representing a total disturbance in the system. That is, the extended state $\boldsymbol{z} = \begin{bmatrix} z_1 & \dots & z_m \end{bmatrix}^T = \begin{bmatrix} \boldsymbol{x}^T & \delta \end{bmatrix}^T \in \mathbb{R}^m$, with $m = n + 1$ is defined with the dynamics given by

$$\begin{aligned} \dot{\boldsymbol{z}} = \boldsymbol{A}_m\boldsymbol{z} + \boldsymbol{d}_m\big(bu + \big(\boldsymbol{\psi}(t, \boldsymbol{\Lambda}_n\boldsymbol{z}) - \boldsymbol{\psi}(t, \boldsymbol{x}_d)\big)\big(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\big) \\ + \boldsymbol{\psi}(t, \boldsymbol{\Lambda}_n\boldsymbol{z})\hat{\boldsymbol{\theta}}\big) + \boldsymbol{b}_m\tfrac{d}{dt}\big(\boldsymbol{\psi}(t, \boldsymbol{x}_d)\big(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\big)\big) \end{aligned} \quad (10)$$

with $\hat{\boldsymbol{\theta}} = \begin{bmatrix} \hat{\theta}_1 & \dots & \hat{\theta}_k \end{bmatrix}^T \in \mathbb{R}^k$ being some estimate of unknown $\boldsymbol{\theta}$. Such a redefinition of the system dynamics allows one to synthesize the PIESO as

$$\dot{\hat{\boldsymbol{z}}} = \boldsymbol{A}_m\hat{\boldsymbol{z}} + \boldsymbol{d}_m\big(bu + \boldsymbol{\psi}(t, \boldsymbol{\Lambda}_n\hat{\boldsymbol{z}})\hat{\boldsymbol{\theta}}\big) + \boldsymbol{l}\big(z_1 - \hat{z}_1\big), \quad (11)$$

where $\hat{\boldsymbol{z}} = \begin{bmatrix} \hat{z}_1 & \dots & \hat{z}_m \end{bmatrix}^T = \begin{bmatrix} \hat{\boldsymbol{x}}^T & \hat{\delta} \end{bmatrix}^T \in \mathbb{R}^m$ is the estimate of the extended state of the system and $\boldsymbol{l} = \begin{bmatrix} l_1 & \dots & l_m \end{bmatrix}^T \in \mathbb{R}_+^m$ are positive observer gains. The control law is proposed according to ADRC paradigm as

$$u = b^{-1}\big(\boldsymbol{k}^T\big(\boldsymbol{x}_d - \boldsymbol{\Lambda}_n\hat{\boldsymbol{z}}\big) - \boldsymbol{\psi}(t, \boldsymbol{\Lambda}_n\hat{\boldsymbol{z}})\hat{\boldsymbol{\theta}} - \hat{z}_m + x_d{}^{(n)}\big) \quad (12)$$

with $\boldsymbol{k} = \begin{bmatrix} k_1 & \dots & k_n \end{bmatrix}^T \in \mathbb{R}_+^n$ being positive controller gains. The proposed control law consists of a simple state feedback term based on the estimates of the state variables, an estimate of the modeled disturbance based on the identified parameters, an estimate of the unknown total disturbance, and the feedforward term. Taking advantage of the representation (10), the total disturbance δ can be interpreted as a measure of the modeling error caused by the parameter uncertainty. This notion can be directly used to formulate an adaptation law as

$$\dot{\hat{\boldsymbol{\theta}}} = \mathrm{Proj}(\boldsymbol{\tau}, \hat{\boldsymbol{\theta}}, \boldsymbol{\Theta}), \quad \boldsymbol{\tau} = \boldsymbol{\Gamma}\boldsymbol{\psi}^T(t, \boldsymbol{\Lambda}_n\hat{\boldsymbol{z}})\hat{z}_m, \quad (13)$$

where $\boldsymbol{\Gamma} \in \mathbb{R}^{k\times k}$ is a positive definite matrix of the adaptation gains to be chosen by the designer and $\mathrm{Proj}(\boldsymbol{\tau}, \hat{\boldsymbol{\theta}}, \boldsymbol{\Theta})$ is a locally Lipschitz projection operator [2], [30]–[32] chosen to satisfy

$$\big(\boldsymbol{\theta}^T - \hat{\boldsymbol{\theta}}^T\big)\boldsymbol{\Gamma}^{-1}\big(\mathrm{Proj}(\boldsymbol{\tau}, \hat{\boldsymbol{\theta}}, \boldsymbol{\Theta}) - \boldsymbol{\tau}\big) \geq 0, \quad (14)$$

and guaranteeing that $\hat{\boldsymbol{\theta}} \in \boldsymbol{\Theta}$ for any $t \geq 0$ and some predefined convex set $\boldsymbol{\Theta}$ such that $\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}(0) \in \boldsymbol{\Theta}$. The notation $\mathrm{Proj}(\boldsymbol{\tau}) = \mathrm{Proj}(\boldsymbol{\tau}, \hat{\boldsymbol{\theta}}, \boldsymbol{\Theta})$ is embraced for brevity in the remaining of the paper.

An application of the projection operator is crucial in scenarios with a regressor depending directly on the state of the system to suppress the influence of potentially large initial estimation or tracking errors on the adaptation procedure. It is employed to disable the adaptation procedure if the estimated regressor or total disturbance are burdened with significant estimation errors which may cause an extensive drift of the parameter estimates. Since the estimates are driven sufficiently close to the true values by virtue of the standard properties of the ESO, the adaptation is renewed to ensure convergence of the parameter estimates.

Corollary 2 (from [28]): By taking advantage of (14) and the definition of a projection operator, it can be shown that there exists some constant $\theta_M \in \mathbb{R}_+$ such that $\max_{t\geq 0}(\|\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\|, \|\hat{\boldsymbol{\theta}}\|) \leq \theta_M$. Thus, it is clear that $\|\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\|^2 \leq \|\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\|\theta_M$. Moreover, $\|\mathrm{Proj}(\boldsymbol{\tau})\| \leq \|\boldsymbol{\tau}\|$ for any $\boldsymbol{\tau}$.

To simplify the choice of observer and controller gains guaranteeing proper behavior of the system, set the gains $\boldsymbol{l}$ and $\boldsymbol{k}$ as $l_i = \bar{l}_i\omega_o^i$ and $k_i = \bar{k}_i\omega_c^{m-i}$ with $\bar{\boldsymbol{l}} = \begin{bmatrix} \bar{l}_1 & \dots & \bar{l}_m \end{bmatrix}^T \in \mathbb{R}^m$ and $\bar{\boldsymbol{k}} = \begin{bmatrix} \bar{k}_1 & \dots & \bar{k}_n \end{bmatrix}^T \in \mathbb{R}^n$ being new scaled gains and $\omega_o, \omega_c \in \mathbb{R}_+$ being new tuning variables called observer and controller bandwidth, respectively [33]. Consider the tracking, estimation, and identification errors given by $\tilde{\boldsymbol{x}} = \boldsymbol{x}_d - \boldsymbol{x}$, $\tilde{\boldsymbol{z}} = \boldsymbol{z} - \hat{\boldsymbol{z}}$, $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta} - \hat{\boldsymbol{\theta}}$. Directly from (5) it follows that $\boldsymbol{\psi}(t, \boldsymbol{x}_d)\tilde{\boldsymbol{\theta}} = \bar{\boldsymbol{\psi}}(t, \boldsymbol{x}_d)\bar{\boldsymbol{\theta}}$, where $\bar{\boldsymbol{\theta}} = \bar{\boldsymbol{\Lambda}}_k\boldsymbol{S}^{-1}\tilde{\boldsymbol{\theta}}$ with $\bar{\boldsymbol{\Lambda}}_k = \begin{bmatrix} \boldsymbol{I}_p & \boldsymbol{0}_{p\times k-p} \end{bmatrix}$. The following theorem can be formulated with respect to the proposed algorithm.

Theorem 1: Given Assumptions 1 and 2 are satisfied, for the system (1), the observer (11) with the control law (12) and the adaptation law (13) guarantees a global asymptotic convergence of $\tilde{\boldsymbol{x}}$, $\tilde{\boldsymbol{z}}$ and $\bar{\boldsymbol{\theta}}$ to the origin, if $\bar{\boldsymbol{l}}$ and $\bar{\boldsymbol{k}}$ gains are chosen such that the matrices $\bar{\boldsymbol{H}} = \boldsymbol{A}_m - \bar{\boldsymbol{l}}\boldsymbol{c}_m^T$ and $\bar{\boldsymbol{G}} = \boldsymbol{A}_n - \boldsymbol{b}_n\bar{\boldsymbol{k}}^T$ are Hurwitz, $\boldsymbol{\Gamma}$ is set with norm small enough, the parameters ω_o, ω_c are chosen high enough, and $\omega_o \geq \omega_c$.

Proof 1: Similarly to the method employed in [28], the stability analysis is henceforth performed in two distinct steps. At first, taking advantage of Corollary 2, the worst-case behavior of tracking and estimation errors is analyzed, noting that the identification errors never exceed the bounds imposed by the projection operator. Then, it is shown that under the adaptation law, the identification errors evolve in such a manner that the performance of the closed-loop system is improved and asymptotic convergence is ensured. Introduce first the scaled tracking and estimation errors in the form of $\bar{x} = \boldsymbol{\Phi}_c\tilde{\boldsymbol{x}}$ and $\bar{\boldsymbol{z}} = \boldsymbol{\Phi}_o\tilde{\boldsymbol{z}}$, where $\boldsymbol{\Phi}_c = \mathrm{diag}(\omega_c^{n-1}, \omega_c^{n-2}, \dots, 1) \in \mathbb{R}^{n\times n}$ and $\boldsymbol{\Phi}_o = \mathrm{diag}(\omega_o^{m-1}, \omega_o^{m-2}, \dots, 1) \in \mathbb{R}^{m\times m}$. The dynamics of the scaled errors are given by

$$\begin{aligned}\dot{\bar{\boldsymbol{x}}} &= \omega_c\bar{\boldsymbol{G}}\bar{\boldsymbol{x}} - \boldsymbol{b}_n\big((\bar{\boldsymbol{k}}^T\omega_c\boldsymbol{\Phi}_c\boldsymbol{\Lambda}_n\boldsymbol{\Phi}_o^{-1} + \boldsymbol{b}_m^T)\bar{\boldsymbol{z}} \\ &\quad + (\boldsymbol{\psi} - \hat{\boldsymbol{\psi}})\hat{\boldsymbol{\theta}} - (\boldsymbol{\psi}_d - \boldsymbol{\psi})\tilde{\boldsymbol{\theta}}\big), \\ \dot{\bar{\boldsymbol{z}}} &= \omega_o\bar{\boldsymbol{H}}\bar{\boldsymbol{z}} + \omega_o\boldsymbol{d}_m\big((\boldsymbol{\psi} - \hat{\boldsymbol{\psi}})\hat{\boldsymbol{\theta}} + (\boldsymbol{\psi} - \boldsymbol{\psi}_d)\tilde{\boldsymbol{\theta}}\big) \\ &\quad + \boldsymbol{b}_m\big(\dot{\boldsymbol{\psi}}_d\tilde{\boldsymbol{\theta}} - \boldsymbol{\psi}_d\,\mathrm{Proj}(\boldsymbol{\tau})\big), \\ \dot{\tilde{\boldsymbol{\theta}}} &= -\mathrm{Proj}(\boldsymbol{\tau}) = -\mathrm{Proj}(\boldsymbol{\Gamma}\hat{\boldsymbol{\psi}}^T\boldsymbol{\psi}_d\tilde{\boldsymbol{\theta}} - \boldsymbol{\Gamma}\hat{\boldsymbol{\psi}}^T\tilde{z}_m),\end{aligned} \tag{15}$$

where $\bar{\boldsymbol{G}} = \boldsymbol{A}_n - \boldsymbol{b}_n\bar{\boldsymbol{k}}^T$ and $\bar{\boldsymbol{H}} = \boldsymbol{A}_m - \bar{\boldsymbol{l}}\boldsymbol{c}_m^T$. Moreover, $\boldsymbol{\psi} = \boldsymbol{\psi}(t, \boldsymbol{\Lambda}_n\boldsymbol{z})$, $\boldsymbol{\psi}_d = \boldsymbol{\psi}(t, \boldsymbol{x}_d)$, $\hat{\boldsymbol{\psi}} = \boldsymbol{\psi}(t, \boldsymbol{\Lambda}_n\hat{\boldsymbol{z}})$ are denoted for brevity. Notably, save for the projection operator, the dynamics of the identification error $\tilde{\boldsymbol{\theta}}$ are in the form reminiscent of the state feedback system with varying gains and external disturbance. The adaptation procedure is thus not directly driven by the output feedback and the commonly featured SPR condition [34] is not necessary in the proposed scheme. Following Theorem 1, gains $\bar{\boldsymbol{k}}$ and $\bar{\boldsymbol{l}}$ are chosen such that $\bar{\boldsymbol{G}}$ and $\bar{\boldsymbol{H}}$ are Hurwitz and thus there exist some positive definite matrices $\boldsymbol{R}$ and $\boldsymbol{P}$ such that $\bar{\boldsymbol{G}}^T\boldsymbol{R} + \boldsymbol{R}\bar{\boldsymbol{G}} = -\boldsymbol{I}_n, \bar{\boldsymbol{H}}^T\boldsymbol{P} + \boldsymbol{P}\bar{\boldsymbol{H}} = -\boldsymbol{I}_m$. Consider the auxiliary function

$$\begin{aligned}V^*(\bar{\boldsymbol{x}}, \bar{\boldsymbol{z}}, \tilde{\boldsymbol{\theta}}) &= \tfrac{1}{2}\omega_o\bar{\boldsymbol{x}}^T\boldsymbol{R}\bar{\boldsymbol{x}} + \tfrac{1}{2}\bar{\boldsymbol{z}}^T\boldsymbol{P}\bar{\boldsymbol{z}} + \tfrac{1}{2}\tilde{\boldsymbol{\theta}}^T\boldsymbol{\Gamma}^{-1}\tilde{\boldsymbol{\theta}} \\ &\quad - \bar{\boldsymbol{z}}^T\boldsymbol{P}\boldsymbol{b}_m\boldsymbol{\psi}(t, \boldsymbol{x}_d)\tilde{\boldsymbol{\theta}}\end{aligned} \tag{16}$$

satisfying

$$\begin{aligned}V^* &\geq \tfrac{1}{2}\omega_o r_m\|\bar{\boldsymbol{x}}\|^2 + \tfrac{1}{2}\big(p_m - \tfrac{1}{\epsilon}\big)\|\bar{\boldsymbol{z}}\|^2 + \tfrac{1}{2}\big(\gamma_M^{-1} \\ &\quad - \epsilon p_M^2\psi_M^2\big)\|\tilde{\boldsymbol{\theta}}\|^2\end{aligned} \tag{17}$$

for any $\epsilon \in \mathbb{R}_+$, where $r_m = \lambda_{\min}(\boldsymbol{R}), p_m = \lambda_{\min}(\boldsymbol{P}) \in \mathbb{R}_+$, and $\gamma_M = \|\boldsymbol{\Gamma}\|, p_M = \|\boldsymbol{P}\| \in \mathbb{R}_+$. By setting $\epsilon > p_m^{-1}$ and choosing $\gamma_M < \big(\epsilon p_M^2\psi_M^2\big)^{-1}$ the positive definiteness of this auxiliary function is ensured. The time derivative of (16) is given by

$$\begin{aligned}\dot{V}^* = &-\tfrac{1}{2}\omega_o\omega_c\bar{\boldsymbol{x}}^T\bar{\boldsymbol{x}} - \omega_o\bar{\boldsymbol{x}}^T\boldsymbol{R}\boldsymbol{b}_n\big((\bar{\boldsymbol{k}}^T\omega_c\boldsymbol{\Phi}_c\boldsymbol{\Lambda}_n\boldsymbol{\Phi}_o^{-1} \\ &+ \boldsymbol{b}_m^T)\bar{\boldsymbol{z}} + (\boldsymbol{\psi} - \hat{\boldsymbol{\psi}})\hat{\boldsymbol{\theta}} - (\boldsymbol{\psi}_d - \boldsymbol{\psi})\tilde{\boldsymbol{\theta}}\big) - \tfrac{1}{2}\omega_o\bar{\boldsymbol{z}}^T\bar{\boldsymbol{z}} \\ &+ \omega_o\bar{\boldsymbol{z}}^T\boldsymbol{P}\boldsymbol{d}_m\big((\boldsymbol{\psi} - \hat{\boldsymbol{\psi}})\hat{\boldsymbol{\theta}} + (\boldsymbol{\psi} - \boldsymbol{\psi}_d)\tilde{\boldsymbol{\theta}}\big) \\ &- \tilde{\boldsymbol{\theta}}^T\boldsymbol{\Gamma}^{-1}\,\mathrm{Proj}(\boldsymbol{\tau}) - \tilde{\boldsymbol{\theta}}^T\boldsymbol{\psi}_d^T\boldsymbol{b}_m^T\boldsymbol{P}\big(\omega_o\bar{\boldsymbol{H}}\bar{\boldsymbol{z}} \\ &+ \omega_o\boldsymbol{d}_m\big((\boldsymbol{\psi} - \hat{\boldsymbol{\psi}})\hat{\boldsymbol{\theta}} + (\boldsymbol{\psi} - \boldsymbol{\psi}_d)\tilde{\boldsymbol{\theta}}\big) \\ &+ \boldsymbol{b}_m\big(\dot{\boldsymbol{\psi}}_d\tilde{\boldsymbol{\theta}} - \boldsymbol{\psi}_d\,\mathrm{Proj}(\boldsymbol{\tau})\big)\big).\end{aligned} \tag{18}$$

On the basis of Assumption 1 the regressor satisfies $\|\boldsymbol{\psi} - \boldsymbol{\psi}_d\| \leq \|\boldsymbol{\Phi}_c^{-1}\|\|\bar{\boldsymbol{x}}\|\psi_M$ and $\|\boldsymbol{\psi} - \hat{\boldsymbol{\psi}}\| \leq \|\boldsymbol{\Lambda}_n\boldsymbol{\Phi}_o^{-1}\|\|\bar{\boldsymbol{z}}\|\psi_M$. Moreover, due to the characteristics of the projection operator, $\|\mathrm{Proj}(\boldsymbol{\tau})\| \leq \gamma_M\big(\psi_M^2\|\boldsymbol{\Lambda}_n\boldsymbol{\Phi}_o^{-1}\|\|\bar{\boldsymbol{z}}\|\|\tilde{\boldsymbol{\theta}}\| + \psi_M^2\|\boldsymbol{\Lambda}_n\boldsymbol{\Phi}_o^{-1}\|\|\bar{\boldsymbol{z}}\|^2 + \psi_M^2\|\boldsymbol{\Phi}_c^{-1}\|\|\bar{\boldsymbol{x}}\|\|\tilde{\boldsymbol{\theta}}\| + \psi_M\|\boldsymbol{\Phi}_c^{-1}\|\|\bar{\boldsymbol{z}}\|\|\bar{\boldsymbol{x}}\| + \psi_M^2\|\tilde{\boldsymbol{\theta}}\| + \psi_M\|\bar{\boldsymbol{z}}\|\big)$. One can now assume, without a loss of generality scaling of $\boldsymbol{l}$ and $\boldsymbol{k}$ such that $\omega_o \geq \omega_c \geq 1$ to ensure that $\|\boldsymbol{\Phi}_c^{-1}\|, \|\boldsymbol{\Lambda}_n\boldsymbol{\Phi}_o^{-1}\|, \omega_o\|\boldsymbol{\Lambda}_n\boldsymbol{\Phi}_o^{-1}\|, \omega_o\|\boldsymbol{\Phi}_c\boldsymbol{\Lambda}_n\boldsymbol{\Phi}_o^{-1}\| \leq 1$.

By recalling property (14) and (4), and finally considering that $\|\tilde{\boldsymbol{\theta}}\| \leq \theta_M$, one can obtain the bound of $\frac{d}{dt}V^*(\bar{\boldsymbol{x}}, \bar{\boldsymbol{z}}, \tilde{\boldsymbol{\theta}})$ as

$$\begin{aligned}\dot{V}^* &\leq \omega_o\big(-\tfrac{1}{2}\omega_c + v_1^* + \tfrac{\epsilon}{2}v_2^*\big)\|\bar{\boldsymbol{x}}\|^2 + \big(\omega_o\big(-\tfrac{1}{2} + \tfrac{1}{2\epsilon}v_2^*\big) \\ &\quad + v_3^*\big)\|\bar{\boldsymbol{z}}\|^2 + \big(v_4^* + \omega_o v_5^*\big)\|\bar{\boldsymbol{z}}\| + \big(v_6^* + \omega_o v_7^*\big)\|\bar{\boldsymbol{x}}\| \\ &\quad + v_8^*\|\bar{\boldsymbol{z}}\|\|\bar{\boldsymbol{x}}\| + v_9^*\end{aligned} \tag{19}$$

for any $\epsilon \in \mathbb{R}_+$, where $k_M = \|\bar{\boldsymbol{k}}\|, h_M = \|\bar{\boldsymbol{H}}\| \in \mathbb{R}_+$ and $v_1^* = r_M\psi_M\theta_M$, $v_2^* = \big(r_M k_M + p_M\psi_M\theta_M + r_M\big), v_3^* = \theta_M p_M\psi_M + \theta_M\psi_M + \theta_M\psi_M^3 p_M\gamma_M, v_4^* = \psi_M^2\theta_M^2 + \theta_M\psi_M + p_M\theta_M^2\psi_M^4\gamma_M + \theta_M^2\psi_M^2 p_M + \theta_M\psi_M^3 p_M\gamma_M, v_5^* = \theta_M\psi_M p_M h_M, v_6^* = p_M\theta_M^2\psi_M^4\gamma_M + \psi_M^2\theta_M^2, v_7^* = \theta_M^2\psi_M^2 p_M, v_8^* = \theta_M\psi_M + \theta_M r_M\psi_M + \theta_M\psi_M^3 p_M\gamma_M, v_9^* = p_M\theta_M^2\psi_M^4\gamma_M + \theta_M^2\psi_M^2 p_M \in \mathbb{R}_+$ are all positive and nonincreasing with growth of ω_o, ω_c or decrease of γ_M. By setting ϵ, ω_c and finally ω_o high enough, this expression can be made negative definite for $\|\bar{\boldsymbol{x}}\|$ and $\|\bar{\boldsymbol{z}}\|$ large enough. Thus, $\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{z}}$ exponentially converge to some boundary of the origin, and there exist some constants $x_M, z_M, T_{xz} \in \mathbb{R}_+$ such that $\|\bar{\boldsymbol{x}}\| \leq x_M$ and $\|\bar{\boldsymbol{z}}\| \leq z_M$ for any $t \geq T_{xz}$. Notably, this conclusion is independent of matrix $\boldsymbol{S}$ or any PE properties of the system.

By taking advantage of the notion of boundedness of the tracking and estimation errors, the convergence of the errors for $t \geq T_{xz}$ can be investigated. Following the approach of [27], [29] one can consider the positive definite function incorporating matrix $\boldsymbol{M}(t, \boldsymbol{x}_d)$ defined as

$$\begin{aligned}V(\bar{\boldsymbol{x}}, \bar{\boldsymbol{z}}, \tilde{\boldsymbol{\theta}}) &= \tfrac{1}{2}\omega_o\bar{\boldsymbol{x}}^T\boldsymbol{R}\bar{\boldsymbol{x}} + \tfrac{1}{2}\bar{\boldsymbol{z}}^T\boldsymbol{P}\bar{\boldsymbol{z}} + \psi_M\tilde{\boldsymbol{\theta}}^T\big(\tfrac{1}{2}\boldsymbol{\Gamma}^{-1} \\ &\quad - \boldsymbol{S}^{-T}\bar{\boldsymbol{\Lambda}}_k^T\boldsymbol{M}(t, \boldsymbol{x}_d)\bar{\boldsymbol{\Lambda}}_k\boldsymbol{S}^{-1}\big)\tilde{\boldsymbol{\theta}}\end{aligned} \tag{20}$$

that satisfies

$$V \geq \tfrac{1}{2}\omega_o r_m\|\bar{\boldsymbol{x}}\|^2 + \tfrac{1}{2}p_m\|\bar{\boldsymbol{z}}\|^2 + \psi_M\big(\tfrac{1}{2}\gamma_M^{-1} - s_m^{-2}\psi_M^2\big)\|\tilde{\boldsymbol{\theta}}\|^2, \tag{21}$$

where $s_m^{-1} = \|\boldsymbol{S}^{-1}\|$. The time derivative of (20) can be written as

$$\begin{aligned}\dot{V} = &-\tfrac{1}{2}\omega_o\omega_c\bar{\boldsymbol{x}}^T\bar{\boldsymbol{x}} - \omega_o\bar{\boldsymbol{x}}^T\boldsymbol{R}\boldsymbol{b}_n\big((\bar{\boldsymbol{k}}^T\omega_c\boldsymbol{\Phi}_c\boldsymbol{\Lambda}_n\boldsymbol{\Phi}_o^{-1} \\ &+ \boldsymbol{b}_m^T)\bar{\boldsymbol{z}} + (\boldsymbol{\psi} - \hat{\boldsymbol{\psi}})\hat{\boldsymbol{\theta}} - (\boldsymbol{\psi}_d - \boldsymbol{\psi})\tilde{\boldsymbol{\theta}}\big) - \tfrac{1}{2}\omega_o\bar{\boldsymbol{z}}^T\bar{\boldsymbol{z}} \\ &+ \omega_o\bar{\boldsymbol{z}}^T\boldsymbol{P}\boldsymbol{d}_m\big((\boldsymbol{\psi} - \hat{\boldsymbol{\psi}})\hat{\boldsymbol{\theta}} + (\boldsymbol{\psi} - \boldsymbol{\psi}_d)\tilde{\boldsymbol{\theta}}\big) \\ &+ \bar{\boldsymbol{z}}^T\boldsymbol{P}\boldsymbol{b}_m\big(\dot{\boldsymbol{\psi}}_d\tilde{\boldsymbol{\theta}} - \boldsymbol{\psi}_d\,\mathrm{Proj}(\boldsymbol{\tau})\big) - \psi_M\tilde{\boldsymbol{\theta}}^T\boldsymbol{\Gamma}^{-1}\,\mathrm{Proj}(\boldsymbol{\tau}) \\ &- 2\psi_M\bar{\boldsymbol{\theta}}^T\boldsymbol{M}(t, \boldsymbol{x}_d)\bar{\boldsymbol{\Lambda}}_k\boldsymbol{S}^{-1}\,\mathrm{Proj}(\boldsymbol{\tau}) \\ &- \psi_M\bar{\boldsymbol{\theta}}^T\boldsymbol{M}(t, \boldsymbol{x}_d)\bar{\boldsymbol{\theta}} + \psi_M\bar{\boldsymbol{\theta}}^T\bar{\boldsymbol{\psi}}_d^T\bar{\boldsymbol{\psi}}_d\bar{\boldsymbol{\theta}}.\end{aligned} \tag{22}$$

and under the notion of the boundedness of the estimation errors and the Lipschitzness of the regressor, for $\omega_o \geq \omega_c \geq 1$, can be bounded by

$$\begin{aligned}\dot{V} &\leq \big(\omega_o\big(-\tfrac{1}{2}\omega_c + v_1 + \tfrac{\epsilon_1}{2}v_2\big) + \tfrac{1}{2}v_3 + \tfrac{\epsilon_3}{2}v_4^2\big)\|\bar{\boldsymbol{x}}\|^2 \\ &\quad + \big(\omega_o\big(-\tfrac{1}{2} + \tfrac{1}{2\epsilon_1}v_2\big) + v_5 + \tfrac{1}{2}v_3 + \tfrac{\epsilon_2}{2}v_6^2\big)\|\bar{\boldsymbol{z}}\|^2 \\ &\quad + \big(-\mu\psi_M^2 e^{-T_{\mathrm{PE}}} + v_7\gamma_M + \tfrac{1}{2\epsilon_2} + \tfrac{1}{2\epsilon_3}\big)\|\bar{\boldsymbol{\theta}}\|^2\end{aligned} \tag{23}$$

for any $\epsilon_1, \epsilon_2, \epsilon_3 \in \mathbb{R}_+$ and $v_1 = r_M\psi_M\theta_M, v_2 = r_M k_M + r_M + p_M\psi_M\theta_M, v_3 = r_M\psi_M\theta_M + p_M\psi_M^3\gamma_M\theta_M + \theta_M\psi_M^2 + 2s_m^{-2}\theta_M\psi_M^4\gamma_M, v_4 = \theta_M\psi_M^3 + 2s_m^{-1}\theta_M\psi_M^5\gamma_M, v_5 = +p_M\psi_M\theta_M + p_M\psi_M^3\gamma_M\theta_M + z_M p_M\psi_M^2\gamma_M + p_M\psi_M^2\gamma_M\big(x_M + 1\big) + \theta_M\psi_M^2 + 2s_m^{-2}\theta_M\psi_M^4\gamma_M, v_6 = p_M\psi_M + \theta_M\psi_M^3 + p_M\psi_M^3\gamma_M + \psi_M^2 + 2s_m^{-1}\theta_M\psi_M^5\gamma_M + 2s_m^{-1}\psi_M^4\gamma_M, v_7 = 2s_m^{-1}\psi_M^5$.

In formula (23) the first terms of the expressions associated with squares of $\|\bar{\boldsymbol{x}}\|$ and $\|\bar{\boldsymbol{z}}\|$ can be made negative by setting ϵ_1 and ω_c large enough. The entire coefficient of squared $\|\bar{\theta}\|$ can be made negative by choosing ϵ_2, ϵ_3 large enough while keeping the norm of $\boldsymbol{\Gamma}$ small enough. Then, the negative definiteness of entire $\dot{V}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{z}}, \bar{\theta})$ can be ensured by choosing ω_o sufficiently large. Notably, function $V(\bar{\boldsymbol{x}}, \bar{\boldsymbol{z}}, \tilde{\theta})$ is positive definite in $\tilde{\theta}$, while $\dot{V}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{z}}, \bar{\theta})$ is negative definite in $\bar{\theta}$ and thus only negative semidefinite in $\tilde{\theta}$. Yet, due to the assumed Lipschitzness of the projection operator the Barbălat lemma [35] can be invoked to conclude about the asymptotic convergence of $\bar{\boldsymbol{x}}, \bar{\boldsymbol{z}}, \bar{\theta}$ to the origin. This concludes the proof.

The presented approach to stability analysis has a straightforward interpretation. As the adaptation law is formulated on the basis of the state and disturbance estimates, in the first step it is ensured that for any initial values these can be brought sufficiently close to their real values by choice of ω_o, ω_c high enough. It is then proved that once the estimation and tracking errors converged to the neighborhood of zero, the adaptation law successfully enforces the convergence of all errors to the origin. Thus, the presented analysis shows that in the first stage the proposed adaptation scheme does not disrupt the well-known property of boundedness of errors in the ADRC, while in the second stage this adaptation scheme additionally improves the performance of the controller. Notably, expressions (17) and (21) feature bandwidth ω_o scaling the terms associated with squares of $\|\bar{\boldsymbol{x}}\|$, while (19) and (23) contain ω_o scaling coefficients of squares of both $\|\bar{\boldsymbol{x}}\|$ and $\|\bar{\boldsymbol{z}}\|$. It can be thus concluded, that for fixed ω_c and $\boldsymbol{\Gamma}$, the growth of $V^*(\bar{\boldsymbol{x}}, \bar{\boldsymbol{z}}, \tilde{\theta})$ and $V(\bar{\boldsymbol{x}}, \bar{\boldsymbol{z}}, \tilde{\theta})$ with increase of the observer bandwidth is outpaced by the growth of absolute values of their respective derivatives. In results, the convergence rate of the errors can be improved by increase of ω_o and the transient performance can thus be enhanced in comparison with standard adaptive methods devoid of active disturbance rejection.

Theorem 1 enables concluding about the asymptotic convergence of the tracking and estimation errors if the standard PE condition is not fully satisfied. Notably, if regressor $\boldsymbol{\psi}(t, \boldsymbol{x}_d)$ do fully satisfy the PE condition on the reference trajectory (i.e. $\boldsymbol{S}$ matrix is identity matrix) it holds that $\bar{\theta} = \tilde{\theta}$ and function V becomes a strict Lyapunov function for the considered system. Global asymptotic and local exponential convergence can be then established for all errors in the system without invoking the Barbălat lemma. Similarly, if the system is entirely missing the excitation (i.e. $\psi_M = 0$) then function V and its derivative become independent of $\tilde{\theta}$ and enables concluding about global exponential convergence of $\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{z}}$ to the origin. The presented analysis thus constitutes a complete framework for the study of proposed method for a wide class of dynamic systems.

While the Theorem considers only the nominal scenario of the system free of structural uncertainties some properties of the algorithm if this condition is not met can be discussed. Specifically, it can be easily shown that the conclusion about the boundedness of the errors drawn on the basis of (16) holds also if the system is affected by some bounded matched Lipschitz disturbance. By taking advantage of this notion and considering function $V_\delta(\bar{\boldsymbol{x}}, \bar{\boldsymbol{z}}, \tilde{\theta}) = \frac{1}{2}\omega_o\bar{\boldsymbol{x}}^T\boldsymbol{R}\bar{\boldsymbol{x}} + \frac{1}{2}\bar{\boldsymbol{z}}^T\boldsymbol{P}\bar{\boldsymbol{z}}$ is can be shown that for the disturbed system the proposed algorithm maintains the key properties of the original ADRC approach. The tracking and estimation errors thus converge to some neighborhood of the origin, which can be made arbitrarily small by increase of ω_c, ω_o. Moreover, by analysis of (20) it can be shown that if the disturbance is small enough $\bar{\theta}$ converges to some neighborhood of the origin, which is smaller that initial bound imposed by the projection operator and improvement of the available estimates of θ is thus achieved. It is worth noting that for the systems subject to constant disturbances the asymptotic convergence can be easily ensured by extending regressor $\boldsymbol{\psi}(t, \boldsymbol{x})$ by additional constant element without any decrease in the performance or applicability of the method.

IV. Simulation validation

The performance of the proposed controller is illustrated by a numerical example. Consider the third-order system in the form of (1) with a regressor given by $\boldsymbol{\psi}(t, \boldsymbol{x}) = \boldsymbol{x}^T$ and parameters $\theta = \begin{bmatrix}-1 & 3 & 3\end{bmatrix}^T$. Such a choice of regressor transforms the system into a standard LTI system, which in the considered case is inherently unstable. The desired trajectory in the form of (2) with $x_d = \sin(\frac{2\pi}{10}t) + \frac{1}{2}\sin(\frac{2\pi}{3}t)$ is defined to ensure persistent excitation of the regressor. The proposed PIDRC is employed with controller and observer bandwidths chosen as $\omega_c = 1, \omega_o = 100$, while the adaptation gain is set to $\boldsymbol{\Gamma} = 0.1\boldsymbol{I}$. The results of the simulation, together with graphs of parameter estimates, Lyapunov function (20) and a comparison with a standard ADRC [33] tuned with the same ω_c, ω_o bandwidths are given in Fig. 1. The function (16) is omitted in the plots since it proved to be meaningful only if extremely large initial errors are present in the system, which is not the case in the considered scenario.

The evolution presented in the plots confirms the superiority of the proposed controller over the classical ADRC approach. The introduction of an adaptive control scheme successfully drives all system errors to the origin, offering a significant improvement over the standard ADRC approach. It can be seen that the parameter estimates converge to their true values, and thus the total disturbance vanishes as the identification progresses. This in turn is a cause of an asymptotic convergence of tracking and state estimation errors. The plots of the Lyapunov function and its derivative confirm the analysis given in the proof of Theorem 1. It is noteworthy that, while the parameter identification process lasts throughout the entire simulation, the tracking and state estimation quality comparable with the standard ADRC approach is achieved almost immediately, as the employed PIESO impacts the disturbance acting upon the plant, but the process of this disturbance estimation is in

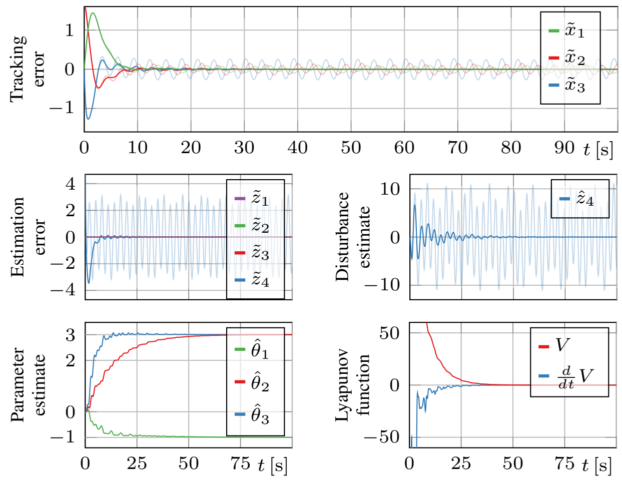

Fig. 1. Results of the numerical simulation of the system with PIDRC controller. The transparent plots present the results produced with the standard ADRC without an adaptation

itself independent of the parameters identification and thus the properties of standard ADRC are maintained regardless of the momentary identification errors.

In order to better visualize the advantages of the proposed solution, a simulation of the system under the classic state feedback adaptive control as proposed in [1] is conducted and compared with the results of the PIDRC algorithm. The feedback gains are chosen the same in both trials and the adaptation gain of the classic solution is selected empirically as $\mathbf{\Gamma} = 0.6\boldsymbol{I}$ to obtain comparable value of the integral of the sum of absolute identification error criterion. Notably, the classic scheme is synthesized under assumption of the full state measurability to simplify the comparison. The results of the simulation are given in Fig. 2. A significant increase in

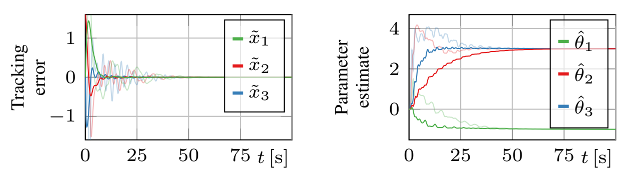

Fig. 2. Results of the numerical simulation of the system with PIDRC controller and classic state feedback solution. Results of the classic algorithm given in transparent plots.

the tracking quality can be noticed in comparison with classic method with much faster vanishing of the transient tracking errors and smaller errors in the initial time instants. Moreover, the higher values of the adaptation gains were required in the classic solution to achieve comparable parameter convergence. The overshoot in the parameter estimates is also reduced by the proposed solution due to the synthesis of the adaptation law directly on the basis of modeling error estimate, although this property is not theoretically guaranteed.

V. Experimental Results

Closed-loop control of the MTracker 3 two-wheeled mobile robot taking advantage of an open hardware and software architecture, [36], is considered for the experimental validation of the proposed algorithm. The robot presented in Fig. 3 is equipped with a TI TMS 320F28335 MCU unit responsible for computation of low-level control procedures. It generates PWM signals to H-bridges which drive two DC motors coupled with pulse encoders that provide the velocity measurements. The dynamics of the robot are expressed by

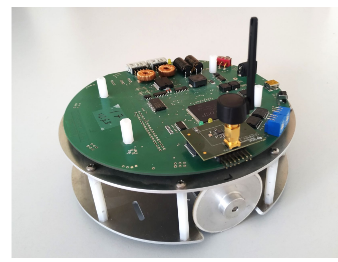

Fig. 3. MTracker 3 robot used for experimental evaluation

$$
\begin{aligned}
\dot{v} &= \beta_v u_v - \tfrac{2\beta}{\bar{m} r^2} v - \tfrac{\mu}{\bar{m} r}\big(\mathrm{sgn}(v + \tfrac{b}{2}\omega) \\
&\quad + \mathrm{sgn}(v - \tfrac{b}{2}\omega)\big) + \tfrac{m l_x}{\bar{m}}\omega^2, \\
\dot{\omega} &= \beta_\omega u_\omega - \tfrac{\beta b^2}{2\bar{J} r^2}\omega - \tfrac{\mu b}{2\bar{J} r}\big(\mathrm{sgn}(v + \tfrac{b}{2}\omega) \\
&\quad - \mathrm{sgn}(v - \tfrac{b}{2}\omega)\big) - \tfrac{m l_x}{\bar{J}}\omega v,
\end{aligned} \tag{24}
$$

where v, ω represent the linear and rotational velocities of the robot expressed in the local frame [37]. Signals u_v, u_ω represent auxiliary inputs in translational and rotational degrees of freedom, while β_v, β_ω stand for the input gains depending on the mass of the robot, its moment of inertia, and some proportional relation between the PWM inputs and the currents of the motors. The mass of the robot is denoted by m, and $\bar{m}, \bar{J}$ are some constants that accumulate mass and inertia together with the effects caused by the offset of the center of mass and the inertia of the wheels. The radius of the robot wheels is expressed as r and the distance between the wheels is given by b. Unknown parameters include β accommodating forces caused by electromagnetic induction and viscous friction, μ representing a Coulomb friction coefficient, and l_x corresponding to the offset of the center of mass from the axis of the wheels. The input signals u_v, u_ω cannot be directly realized, but by taking advantage of the geometric properties of the robot, they can be reformulated as $u_v = \frac{1}{r}(u_R + u_L), u_\omega = \frac{b}{2r}(u_R - u_L)$, where u_R, u_L correspond to the PWM signals of the motors of the right and left wheels. Equation (24) can be rewritten as two coupled systems roughly corresponding to the assumed model (1). Thus, the dynamics of the robot are expressed as

$$
\dot{v} = \beta_v u_v + \boldsymbol{\psi}_v(v, \omega)\boldsymbol{\theta}_v, \quad \dot{\omega} = \beta_\omega u_\omega + \boldsymbol{\psi}_\omega(\omega, v)\boldsymbol{\theta}_\omega, \tag{25}
$$

with parameter vectors given as $\boldsymbol{\theta}_v = \bar{m}^{-1}\boldsymbol{\theta}, \boldsymbol{\theta}_\omega = \bar{J}^{-1}\boldsymbol{\theta}$ and $\boldsymbol{\theta} = \begin{bmatrix}\beta & \mu & l_x\end{bmatrix}^T$. Moreover, the regressors of each subsystem are given by $\boldsymbol{\psi}_v = \begin{bmatrix}-\frac{2v}{r^2} & -\frac{1}{r}\big(\mathrm{sgn}(v + \frac{b\omega}{2}) + \mathrm{sgn}(v - \frac{b\omega}{2})\big) & m\omega^2\end{bmatrix}$ and $\boldsymbol{\psi}_\omega = -\begin{bmatrix}\frac{b^2\omega}{2r^2} & \frac{b}{2r}\big(\mathrm{sgn}(v + \frac{b\omega}{2}) - \mathrm{sgn}(v - \frac{b\omega}{2})\big) & m\omega v\end{bmatrix}$. For each of the systems (25) the PIDRC controller is designed according to (11)–(13). To this end, the known parameters are set to $\beta_v = 0.06, \beta_\omega = 13.85$. The controller gain is chosen as

$k = 7.5$ and the observer gains are set as $l_1 = 5, l_2 = 6.25$ for both subsystems. The desired linear and rotational velocities are generated online in an open loop. In the experiments the matrices $\mathbf{\Gamma}$ are chosen to be in diagonal form with elements on the diagonal given by $\gamma_{v1} = \frac{4}{10^8}, \gamma_{v2} = \frac{2}{10^6}, \gamma_{v3} = \frac{2}{10^4}$ and $\gamma_{\omega 1} = \frac{2}{10^5}, \gamma_{\omega 2} = \frac{4}{10^4}, \gamma_{\omega 3} = 2$. The results of the experiment are given in Fig. 4, where the tracking errors, the disturbance estimates, and the parameter estimates are given. A significant improvement in the tracking quality, measured in terms of tracking errors, is observed in both subsystems of the robot. To quantify obtained results, the mean integral of the square error criteria are calculated as $E_v = \frac{1}{t_f}\int_0^{t_f} \tilde{v}^2 dt$ and $E_\omega = \frac{1}{t_f}\int_0^{t_f} \tilde{\omega}^2 dt$, with t_f being the final time of the experiment. Values of $E_v \approx 5.2 \cdot 10^{-3}$ and $E_\omega \approx 1.32$ are obtained for standard ADRC controller, while the proposed PIDRC approach resulted in $E_v \approx 8.9 \cdot 10^{-4}$ and $E_\omega \approx 0.2$, what corresponds to decrease of the considered quality factor by over 82% in translational subsystem and over 84% in rotational subsystem.

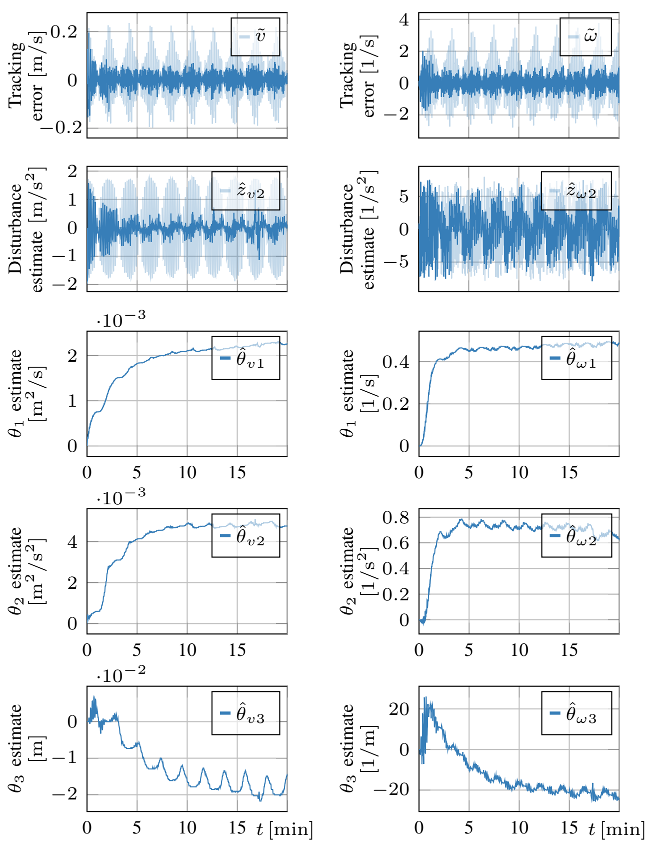

Fig. 4. Results of the experiment with the mobile robot and PIDRC controller based on the estimated states. Transparent plots present results obtained with standard non-adaptive ADRC.

The vanishing of the disturbance estimate is visible in the translational subsystem. A similar observation in the second subsystem is hindered by the presence of stronger measurement noises and external disturbances. Although external disturbances other than modeling error due to parameter uncertainty are not accounted for in the theoretical analysis of the method, the improved tracking quality is maintained due to the inherent ability of the ADRC scheme to cope with unmodeled disturbances. Specifically, the disturbance estimates produced in the experiment correspond to the sum of modeling errors and unaccounted external disturbances. While the adaptation procedure is designed to cope with the former of the two, in the steady state the latter is compensated by the direct presence of the disturbance estimate in the control law (12). Meanwhile, the convergence of the parameter estimates associated with both degrees of freedom to some constant values is displayed. According to (25) the results of the identification can be evaluated by calculating the estimate of $\frac{\boldsymbol{\theta}_v}{\boldsymbol{\theta}_\omega} = \frac{\bar{J}}{\bar{m}}$ across all parameters. In this regard the performed experiment results in $\frac{\hat{\theta}_{v1}}{\hat{\theta}_{\omega 1}} \approx 0.0047$, $\frac{\hat{\theta}_{v2}}{\hat{\theta}_{\omega 2}} \approx 0.0076$ and $\frac{\hat{\theta}_{v3}}{\hat{\theta}_{\omega 3}} \approx 0.0006$. One can state that a satisfactory quality of identification is obtained for the first and second parameters for both subsystems. The quality of identification of the third parameter is probably compromised by the possible presence of some unmodeled disturbances that affect the plant or insufficient excitation of the regressor. Although the identification of some of the parameters can be deemed slow, the decrease in the tracking error is obtained after a relatively short time due to the properties of the employed ADRC-based disturbance rejection scheme.

VI. Conclusions

In this paper, a new adaptive control scheme based on the ADRC approach was introduced. The proposed algorithm of Parameter Identifying Disturbance Rejection Control guarantees the asymptotic convergence of tracking, estimation, and identification errors for controllable systems linear in parameters. Importantly, this algorithm inherits the advantageous characteristics of the ADRC method, enabling a fast convergence of tracking errors to a neighborhood of zero, as well as an adaptive approach that gradually reduces these errors to zero. This enhancement in control performance compared to the classical ADRC structure has been proved analytically using the Lyapunov approach and observed through both simulation and experimental studies.

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