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Scalar-Measurement Attitude Estimation on $\mathrm{SO}(3)$ with Bias Compensation

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***Abstract*— Attitude estimation methods typically rely on full vector measurements from inertial sensors such as accelerometers and magnetometers. This paper shows that reliable estimation can also be achieved using only scalar measurements, which naturally arise either as components of vector readings or as independent constraints from other sensing modalities. We propose nonlinear deterministic observers on $\mathbf{SO}(3)$ that incorporate gyroscope bias compensation and guarantee uniform local exponential stability under suitable observability conditions. A key feature of the framework is its robustness to partial sensing: accurate estimation is maintained even when only a subset of vector components is available. Experimental validation on the BROAD dataset confirms consistent performance across progressively reduced measurement configurations, with estimation errors remaining small even under severe information loss. To the best of our knowledge, this is the first work to establish fundamental observability results showing that two scalar measurements under suitable excitation suffice for attitude estimation, and that three are enough in the static case. These results position scalar-measurement-based observers as a practical and reliable alternative to conventional vector-based approaches.**

***Index Terms*— Uniform Observability, Scalar Measurements, Observers for Nonlinear Systems, Continuous Riccati Equation.**

I. INTRODUCTION

Accurate attitude (orientation) estimation is a fundamental requirement in diverse applications such as spacecraft stabilization, aerial and ground vehicle control, and autonomous navigation [1]. To achieve this, reliable state observers (filters) have been extensively studied, aiming to fuse information from multiple sensors—typically providing vector inertial measurements, expressed in either the inertial or body frame, together with body-frame angular velocity measurements [2]–[6]. The most widely used sensors for attitude estimation are Inertial Measurement Units (IMUs), which typically integrate tri-axial accelerometers and gyroscopes, and may also include a magnetometer. Despite their low cost and compactness, IMUs suffer from limited accuracy and high sensitivity to noise. In particular, gyroscopes—crucial for observer design due to their high bandwidth—are affected by drift and bias [7].

Early solutions to Wahba's problem [8] focused on deterministic reconstruction methods, such as Davenport's q-method, Shuster's QUEST algorithm, and Markley's SVD-based approach [9], [10]. These techniques provide closed-form or iterative solutions to the attitude estimation problem from vector observations, but they do not explicitly handle sensor noise or dynamic models. To address these aspects, Kalman-type filters were later introduced for attitude and bias estimation [7]. However, these filters are computationally intensive and rely on linear approximations, requiring careful tuning and implementation [1]. To overcome such limitations, invariant Kalman filters have emerged as a robust alternative, offering local asymptotic stability and improved performance in nonlinear settings [11], [12]. In parallel, nonlinear deterministic observers have been developed, offering stronger theoretical stability guarantees and better handling of the nonlinear dynamics inherent to Inertial Navigation Systems (INS) [2], [4], [13]–[16]. While these approaches have shown promising results, they typically assume the availability of complete 3D vector measurements–an assumption that may not hold in practice [17].

This paper revisits the scalar-measurement framework for rigid-body orientation estimation first introduced in [17]. Such measurements are often expressed as the cosine of the angle between two vectors, but they may also arise from alternative constraints, including tilt relations obtained from barometer and range sensors or landmark-based altitude differences. Unlike full vector observations, scalar measurements provide only partial constraints on the attitude, yet they enable a more flexible and robust estimation process, particularly in scenarios where directional data are sparse, noisy, or partially unavailable. The formulation in [17], however, relied on embedding $\mathbf{SO}(3)$ into $\mathbb{R}^9$, which imposed unnecessarily strong uniform observability conditions and did not address bias. To overcome these limitations, we adopt a direct design on $\mathbf{SO}(3)$ that explicitly incorporates gyroscope bias. Building on the Riccati-based observer framework of [18], we propose a deterministic filter inspired by the multiplicative Extended Kalman Filter (MEKF), where the continuous Riccati equation (CRE) governs both observer dynamics and Lyapunov-based stability analysis [19].

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The main contributions are fourfold: (i) a direct $\mathbf{SO}(3)$-based observer that avoids high-dimensional embeddings, (ii) explicit bias handling within a deterministic Riccati framework, (iii) rigorous persistence-of-excitation conditions linking observability of the linearized dynamics to local exponential stability, and (iv) new fundamental insights into scalar-based estimation. In particular, while it is well known that at least two non-collinear vector measurements are required to reconstruct attitude [2], [3], we show that two scalar measurements under suitable excitation suffice for attitude observability, and that three scalar measurements are enough in the static case, matching the intrinsic three-dimensional nature of $\mathbf{SO}(3)$. To the best of our knowledge, these results have not been reported before. Finally, the proposed approach is validated experimentally under partial and noisy measurements.

The remainder of this paper is organized as follows. Section II introduces the notation, system equations, and measurement models, along with essential concepts on uniform observability and the CRE-based observer. Section III presents the observer design for both unbiased and biased angular velocity measurements, together with conditions ensuring local exponential stability. Section IV reports experimental results, and Section V concludes the paper.

II. Preliminary Material

A. Notation

- $\mathcal{I} = \{G_{\mathcal{I}}, e_1, e_2, e_3\}$ denotes a right-handed inertial reference frame with fixed origin $G_{\mathcal{I}}$ and standard basis vectors of $\mathbb{R}^3$. The body-fixed frame $\mathcal{B} = \{G_{\mathcal{B}}, e_{B1}, e_{B2}, e_{B3}\}$ is attached to the vehicle, with its origin $G_{\mathcal{B}}$ at the center of mass.
- The Euclidean norm of a vector $x \in \mathbb{R}^n$ is denoted by $|x|$. The unit 2-sphere is represented as $\mathbb{S}^2 := \{v \in \mathbb{R}^3 \mid |v| = 1\}$. The set $\mathfrak{B}_r^n := \{x \in \mathbb{R}^n \mid |x| \le r\}$ denotes the closed ball in $\mathbb{R}^n$ of radius r.
- For any vector $\Omega \in \mathbb{R}^3$, $\Omega^\times$ is the skew-symmetric matrix associated with the cross product, satisfying $\Omega^\times y = \Omega \times y$ for all $y \in \mathbb{R}^3$.
- The special orthogonal group, denoted as $\mathbf{SO}(3)$, is the Lie group of 3D rotations given by $\mathbf{SO}(3) := \{R \in \mathbb{R}^{3\times 3} \mid R^\top R = RR^\top = I_3, \det(R) = 1\}$. Its associated Lie algebra is defined as $\mathfrak{so}(3) := \{\Omega^\times \mid \Omega \in \mathbb{R}^3\}$.
- Let f be a vector-valued function of two variables x and y, and time t. We write $f = O(|x|^{k_1}|y|^{k_2})$, with $k_1 \ge 0, k_2 \ge 0$, if for all t: $|f(x,y,t)|/(|x|^{k_1}|y|^{k_2}) \le \gamma < \infty$ in the vicinity of $(x = 0, y = 0)$.

For clarity and conciseness, the argument of time-dependent signals is provided when necessary and omitted otherwise.

B. Observability Principles and Requirements

Consider a generic linear time-varying (LTV) system:

$$\dot{\boldsymbol{x}} = A(t)\boldsymbol{x} + B(t)\boldsymbol{u}, \qquad \boldsymbol{y} = C(t)\boldsymbol{x}, \tag{1a}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ denotes the state, $\boldsymbol{u} \in \mathbb{R}^s$ the input, and $\boldsymbol{y} \in \mathbb{R}^m$ the output. The following definition of observability related to this system is derived from the works [20], [21].

Definition 1 (Uniform Observability): The system (1) is said to be *uniformly observable* if there exist constants $\delta, \mu > 0$ such that, for all $t \ge 0$:

$$W(t, t+\delta) := \frac{1}{\delta}\int_t^{t+\delta} \Phi^\top(s,t)C^\top(s)C(s)\Phi(s,t)\,ds \succeq \mu I_n, \tag{2}$$

where $W(t, t+\delta)$ is the observability Gramian associated with the pair $(A(t), C(t))$, and $\Phi(s,t)$ denotes the state transition matrix associated with $A(t)$, defined by: $\frac{d}{dt}\Phi(s,t) = A(t)\Phi(s,t)$, $\Phi(t,t) = I_n$. If condition (2) holds, the pair $(A(t), C(t))$ is said to be *uniformly observable*. □

C. Riccati Observers for a Class of Nonlinear Systems

Consider a class of nonlinear systems with state $\boldsymbol{x} := (x_1, x_2) \in \mathfrak{B}_r^{n_1} \times \mathbb{R}^{n_2}$, $n = n_1 + n_2$, input $\boldsymbol{u} \in \mathbb{R}^s$, and output $\boldsymbol{y} \in \mathbb{R}^m$ satisfying the dynamics

$$\dot{\boldsymbol{x}} = A(x_1,t)\boldsymbol{x} + \boldsymbol{u} + O(|x_1||\boldsymbol{u}|) + O(|x_1||x_2|), \tag{3a}$$
$$\boldsymbol{y} = C(x_1,t)\boldsymbol{x} + O(|x_1|^2), \tag{3b}$$

where the matrix-valued functions $A(x_1,t) \in \mathbb{R}^{n\times n}$ and $C(x_1,t) \in \mathbb{R}^{m\times n}$ are continuous matrix-valued functions uniformly bounded w.r.t. t. The following result is adapted from Theorem 3.1 and Corollary 3.2 in [18].

Proposition 1: Consider the system dynamics (3) and choose the input signal:

$$\boldsymbol{u} = -P(t)C^\top(x_1,t)Q(t)\boldsymbol{y}, \tag{4}$$

with $P(0)$ a positive definite (p.d.) matrix solution to the Continuous Riccati Equation (CRE)

$$\dot{P} = AP + PA^\top - PC^\top Q(t)CP + V(t), \tag{5}$$

with $Q(t)$ and $V(t)$ bounded continuous symmetric positive definite matrix-valued functions. If the pair $(A^\star(t), C^\star(t)) := (A(0,t), C(0,t))$ is uniformly observable, then the origin of (3) is locally exponentially stable. □

The result follows from standard Riccati-based Lyapunov arguments. Since $A(x_1,t)$ and $C(x_1,t)$ depend only on x_1, compactness of $\mathfrak{B}_r^{n_1}$ ensures their uniform boundedness along trajectories. This, combined with uniform observability of the pair $(A^\star(t), C^\star(t))$, guarantees boundedness and well-conditioning of $P(t)$, from which local exponential stability follows. See Theorem 3.1 in [18] for details (cf. [22]).

D. System Equations and Measurements

Let $R \in \mathbf{SO}(3) : \mathcal{B} \to \mathcal{I}$ denote the orientation of a moving body, mapping the body-fixed frame $\mathcal{B}$ to the inertial frame $\mathcal{I}$. The attitude evolves according to $\dot{R} = R\Omega^\times$, where $\Omega \in \mathbb{R}^3$ is the angular velocity, typically provided by a tri-axial gyroscope. In practice, the gyro measurement takes the form $\Omega_y = \Omega + d + \mu_\Omega$ where the true body-frame angular velocity is corrupted by an additive noise term μ_Ω and a slowly time-varying bias d. Since we focus on the design of a deterministic observer, we idealize the measurement by assuming noise-free gyros ($\mu_\Omega \equiv 0$), and adopt the following model to describe the dynamics of the system of interest:

$$\dot{R} = R(\Omega_y - d)^\times, \qquad \dot{d} = 0. \tag{6}$$

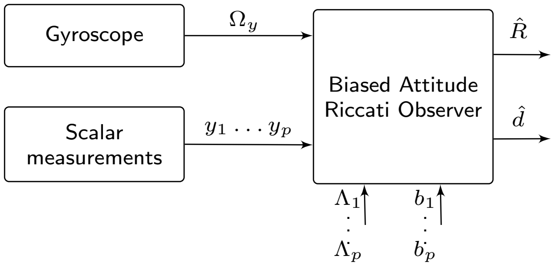

Fig. 1: Illustration of the proposed estimation approach.

As for the output measurements, the moving object is assumed to be equipped with a suite of sensors located at the origin of $\mathcal{B}$, providing the outputs $y_i \in \mathbb{R}^{n_i}$, $i = 1, \cdots, p$, defined as

$$y_i := \Lambda_i^\top R^\top b_i, \qquad \Lambda_i := \begin{bmatrix} a_1^{(i)} & \cdots & a_{n_i}^{(i)} \end{bmatrix}, \tag{7}$$

which compose the full output measurement vector

$$y := [y_1^\top \ \cdots \ y_p^\top]^\top \in \mathbb{R}^m,$$

with $m = \sum_{i=1}^p n_i$. Each y_i represents the set of scalar outputs associated with the possibly time-varying known inertial direction $b_i \in \mathbb{R}^3$, collected by measuring b_i along the body-frame directions $a_j^{(i)} \in \mathbb{R}^3$, $j = 1, \cdots, n_i$, $n_i \in \mathbb{N}/\{0\}$. In the case of a classical IMU, let $R_{\mathrm{IMU}} \in \mathbf{SO}(3)$ denote the constant alignment between the IMU sensor axes and the body frame $\mathcal{B}$. Assuming inertial acceleration is negligible compared to gravity, the accelerometer provides three scalar measurements collected in the output $y_1 = \Lambda_1^\top R^\top b_1$ with $\Lambda_1 \in \mathbb{R}^{3\times 3}$ composed by $a_j^{(1)} = R_{\mathrm{IMU}} e_j$, for $j = 1, 2, 3$, and $b_1 = e_3$, where e_3 is the gravity direction in the inertial frame. Similarly, the magnetometer channels satisfy the same model with $a_j^{(2)} = R_{\mathrm{IMU}} e_j$, for $j = 1, 2, 3$, and $b_2 = m_0$, where m_0 denotes the magnetic field direction in the inertial frame. Further examples of scalar attitude measurements derived from other sensors can be found in [17].

III. Observer Design

By exploiting the above modeling (II-D)-(7), the proposed observer equation is posed directly as a kinematic system for an attitude estimate on $\mathbf{SO}(3)$ along with an estimate of the bias in $\mathbb{R}^3$. The observer kinematics consist of a prediction term based on the measurement Ω_y, and an innovation term $\Delta = (\Delta_R, \Delta_d) \in \mathbb{R}^6$ derived from the available measurements through the Riccati observer framework described in [18]. The proposed attitude estimation approach is illustrated in Fig. 1. The general form proposed for the observer is:

$$\dot{\hat{R}} = \hat{R}(\Omega_y - \hat{d})^\times + \Delta_R^\times \hat{R}, \quad \hat{R}(0) = \hat{R}_0, \tag{8a}$$

$$\dot{\hat{d}} = -\Delta_d, \quad \hat{d}(0) = \hat{d}_0. \tag{8b}$$

Let $\tilde{d} = d - \hat{d} \in \mathbb{R}^3$ denote the bias error, and define the attitude error on $\mathbf{SO}(3)$ by $\tilde{R} = \hat{R}R^\top$. Combining the true and estimated dynamics yields the nonlinear error dynamics:

$$\dot{\tilde{R}} = \left(\hat{R}\tilde{d} + \Delta_R\right)^\times \tilde{R}, \qquad \dot{\tilde{d}} = \Delta_d. \tag{9}$$

The output error is defined as

$$\tilde{y}_i := \hat{y}_i - y_i = \Lambda_i^\top \hat{R}^\top b_i - \Lambda_i^\top R^\top b_i = \Lambda_i^\top \hat{R}^\top (I_3 - \tilde{R}) b_i. \tag{10}$$

To derive a first-order approximation of the error dynamics (9), we exploit a local compact parametrization of $\mathbf{SO}(3)$ via the unit quaternion $\tilde{Q} := (\tilde{q}_0, \tilde{q}) \in \mathbb{S}^3$ corresponding to the rotation error $\tilde{R}$. Rodrigues' formula writes

$$\tilde{R} = I_3 + 2\tilde{q}^\times(\tilde{q}_0 I_3 + \tilde{q}^\times), \tag{11}$$

from which a first-order approximation of $\tilde{R}$ around I_3 is obtained as

$$\tilde{R} = I_3 + \tilde{\lambda}^\times + O(|\tilde{\lambda}|^2), \tag{12}$$

with $\tilde{\lambda} := 2\,\mathrm{sign}(\tilde{q}_0)\tilde{q} \in \mathfrak{B}_2^3$. It follows that the observer error dynamics can be expressed as

$$\dot{\tilde{\lambda}} = \hat{R}\tilde{d} + \Delta_R + O(|\tilde{\lambda}||\Delta_R|) + O\big(|\tilde{\lambda}||\tilde{d}|\big), \tag{13a}$$

$$\dot{\tilde{d}} = \Delta_d, \tag{13b}$$

while the output error becomes

$$\tilde{y}_i = \Lambda_i^\top \hat{R} b_i^\times \tilde{\lambda} + O(|\tilde{\lambda}|^2). \tag{14}$$

By setting $\boldsymbol{x} := (\tilde{\lambda}, \tilde{d}) \in \mathfrak{B}_2^3 \times \mathbb{R}^3$, $\boldsymbol{y} := (\tilde{y}_1, \cdots, \tilde{y}_p) \in \mathbb{R}^m$, and $\boldsymbol{u} := \Delta \in \mathbb{R}^6$, the error dynamics (13) with outputs (14) can be rewritten in the form (3), with

$$A(\tilde{\lambda}, t) := \begin{bmatrix} 0_{3,3} & \hat{R} \\ 0_{3,3} & 0_{3,3} \end{bmatrix}, \ C(\tilde{\lambda}, t) := \begin{bmatrix} C_1(\tilde{\lambda}, t) & 0_{1,3} \\ \vdots & \vdots \\ C_p(\tilde{\lambda}, t) & 0_{1,3} \end{bmatrix},$$

with $C_i(\tilde{\lambda}, t) := \Lambda_i^\top \hat{R}^\top b_i^\times$, $i = 1, \ldots, p$. In light of (4), we choose the innovation $\Delta = -PC^\top Q\boldsymbol{y}$, where $P(t)$ is the solution to the CRE (5) associated with (13). According to Proposition 1, the exponential stability of the equilibrium $\boldsymbol{x} = 0$ is related to the uniform observability of the pair $(A^\star(t) := A(0, t), C^\star(t) := C(0, t))$, given by

$$A^\star(t) := \begin{bmatrix} 0_{3,3} & R \\ 0_{3,3} & 0_{3,3} \end{bmatrix}, \quad C^\star(t) := \begin{bmatrix} C_1^\star(t) & 0_{1,3} \\ \vdots & \vdots \\ C_p^\star(t) & 0_{1,3} \end{bmatrix}, \tag{15}$$

with $C_i^\star(t) := \Lambda_i^\top R^\top b_i^\times$, $i = 1, \ldots, p$. It therefore remains to provide explicit conditions under which uniform observability of $(A^\star(t), C^\star(t))$ is guaranteed.

A. Unbiased case

To simplify the analysis, we first consider the unbiased scenario, *i.e.,* $d \equiv 0$. This allows us to separate the analysis and treat the simpler attitude-only system first, then extend to the full biased problem. In this case, system (13) reduces to the attitude error $\tilde{\lambda}$ alone:

$$\dot{\tilde{\lambda}} = \Delta_R + O(|\tilde{\lambda}||\Delta_R|) + O(|\tilde{\lambda}|^2), \tag{16a}$$

$$\boldsymbol{y} = C_\lambda(\tilde{\lambda}, t)\tilde{\lambda} + O(|\tilde{\lambda}|^2), \tag{16b}$$

where $A_\lambda := 0$, and $C_\lambda(\tilde{\lambda}, t) := [C_1^\top, \ldots, C_p^\top]^\top$. It follows directly that $A_\lambda^\star \equiv A_\lambda$ and $C_\lambda^\star(t) := [C_1^\star, \ldots, C_p^\star]^\top$.

The following lemma is a direct application of Proposition 1, leveraging the definition of observability Gramian of the pair $(A^\star_\lambda, C^\star_\lambda(t))$ to establish local exponential stability of the attitude error $\tilde{R}$, which will be instrumental for the stability analysis of the full error $(\tilde{R}, \tilde{d})$.

Lemma 1: Suppose that the input signal Ω and the measurement directions $a_j^{(i)}, b_i$ are continuous and bounded, and the observability Gramian

$$W^{A^\star_\lambda, C^\star_\lambda}(t, t+\delta) := \frac{1}{\delta}\int_t^{t+\delta} \sum_{i=1}^{m} b_i^\times R(s)\Lambda_i\Lambda_i^\top R^\top(s)\, b_i^\times \, ds, \tag{17}$$

satisfies (2). Then, the equilibrium $\tilde{R} = I_3$ is locally exponentially stable. □

This result guarantees local convergence while relaxing the sufficient condition of Lemma 1 in [17]. In fact, the linear formulation in $\mathbb{R}^9$ imposes a stricter uniform observability requirement, whereas the intrinsic formulation on $\mathbf{SO}(3)$ requires uniform observability only in three dimensions. Consequently, this result strengthens Lemma 1 in [17] by establishing a necessary and sufficient uniform observability condition that can be applied across different scenarios.

To illustrate this, consider the classical and simplest case involving constant directions $a_j^{(i)}, b_i \in \mathbb{S}^2$. The next corollary provides explicit conditions for uniform observability of $(A^\star_\lambda, C^\star_\lambda(t))$ for the case of two and three scalar measurements ($m = 2, 3$) with two non-collinear inertial directions b_1 and b_2, which, *e.g.*, might represent the inertial directions available from the pair accelerometer-magnetometer. Notice how $m = 3$ is the minimum number of scalar measurements that guarantees uniform observability for R constant.

Definition 2: For any vector $\alpha(t) \in \mathbb{R}^3$, define

$$U_{t,\delta}(\alpha) = \frac{1}{\delta}\int_t^{t+\delta} \alpha(s)\alpha^\top(s)\, ds, \tag{18}$$

with eigenvalues $\lambda_1(t,\delta) \le \lambda_2(t,\delta) \le \lambda_3(t,\delta)$. Then $\alpha(t)$ is called *strongly persistently exciting* if there exist $\delta, \beta > 0$ such that $\lambda_1(t,\delta) \ge \beta$ for all $t \ge 0$. It is *weakly persistently exciting* if $\lambda_2(t,\delta) \ge \beta$. If $\mathrm{rank}(U_{t,\delta}(\alpha)) = 1$, then $\alpha(t)$ cannot be persistently exciting. □

Corollary 1: Let $a_j^{(i)}, b_i \in \mathbb{S}^2$ in (7) be constant known vectors. We distinguish two cases:

1) If $m = 2$ with two non-collinear inertial directions b_1, b_2, i.e. $\Lambda_1 = a_1^{(1)}, \Lambda_2 = a_1^{(2)}$, then uniform observability is guaranteed if either:
   i) $R\Lambda_1$ and $R\Lambda_2$ are strongly persistently exciting.
   ii) Only $R\Lambda_1$ is strongly persistently exciting, and there exist $\delta, \mu > 0$ such that
   $$\frac{1}{\delta}\int_t^{t+\delta} \det([R(s)\Lambda_2 \ b_1 \ b_2])^2 ds > \mu, \tag{19}$$
   or equivalently, only $R\Lambda_2$ is strongly persistently exciting and $R\Lambda_1$ satisfies (19).
   iii) $R\Lambda_2$ and $R\Lambda_2$ are weakly persistently exciting and there exist $\delta, \mu > 0$ such that for $i \in \{1, 2\}$:
   $$\frac{1}{\delta}\int_t^{t+\delta} \det([R(s)\Lambda_i \ b_1 \ b_2])^2 ds > \mu. \tag{20}$$
2) If $m = 3$ with two non-collinear inertial directions b_1, b_2 and $\Lambda_1 = \begin{bmatrix} a_1^{(1)} & a_2^{(1)} \end{bmatrix}, \Lambda_2 = a_1^{(2)}$, then uniform observability is guaranteed if either:
   i) there exist $\delta, \mu > 0$ such that
   $$\frac{1}{\delta}\int_t^{t+\delta} \det([R(s)\Lambda_1 \ b_1])^2 ds > \mu, \tag{21a}$$
   $$\frac{1}{\delta}\int_t^{t+\delta} \det([R(s)\Lambda_2 \ b_1 \ b_2])^2 ds > \mu. \tag{21b}$$
   i*) Assuming R constant,
   $$\det([R\Lambda_1 \ b_1]) \neq 0, \quad \det([R\Lambda_2 \ b_1 \ b_2]) \neq 0.$$

□

The proof, reported in [22], follows by direct inspection of the observability Gramian (17), which, for constant $a_j^{(i)}, b_i \in \mathbb{S}^2$, can be expressed as

$$W^{A^\star_\lambda, C^\star_\lambda}(t, t+\delta) = -\sum_{i=1}^{p} b_i^\times \left(\sum_{j=1}^{n_i} U_{t,\delta}(Ra_j^{(i)})\right) b_i^\times.$$

B. Biased case

We now extend the stability analysis to system (13), where the bias must also be estimated. The condition established in Lemma 1 is a prerequisite to guarantee the stability of the attitude subsystem (16). Building on this, we turn to the full error dynamics $\boldsymbol{x}$. The following lemma provides a sufficient condition for the uniform observability of $(A^\star(t), C^\star(t))$.

Lemma 2: Suppose that the input signal Ω is continuous and bounded, that the measurement directions $a_j^{(i)}, b_i$ in (7) are uniformly continuous and bounded, and that condition (17) is verified. Moreover, assume that there exist constants $\delta, \mu > 0$ such that, for all $t \ge 0$,

$$\begin{aligned} &W^{\bar{C}^\star}(t, t+\delta) \\ &\quad := \frac{1}{\delta}\int_t^{t+\delta} \sum_{i=1}^{m} \bar{C}_i^{\star\top}(s,t,\delta)\bar{C}_i^\star(s,t,\delta) ds \succeq \mu I_3, \end{aligned} \tag{22}$$

where $\bar{C}_i^\star(s,t,\delta) := C_i^\star(s)\left(\int_t^s R(\tau)d\tau - \rho(t,\delta)\right)$, and

$$\rho(t,\delta) := \left(W^{A^\star_\lambda, C^\star_\lambda}\right)^{-1} \int_t^{t+\delta} \sum_{i=1}^{m} C_i^{\star\top} C_i^\star \int_t^s R(\tau)d\tau ds.$$

Consequently, the pair $(A^\star(t), C^\star(t))$ is uniformly observable and $(\tilde{R}, \tilde{d}) = (I_3, 0)$ is locally exponentially stable. □

The proof, provided in [22], is based on a Schur complement argument that links conditions (17)–(22) to the uniform observability of $(A^\star(t), C^\star(t))$. The result then follows from Proposition 1. In general, condition (22) is difficult to verify for time-varying vectors $a_j^{(i)}, b_i$. However, when the vectors $a_j^{(i)}, b_i$ are constant and mild additional assumptions are imposed, the condition simplifies considerably. The following corollary gives *sufficient conditions* in the case $m = p = 2$ and $\Lambda_1 = \Lambda_2$. All other cases can be understood as relaxations of the conditions in this corollary.

Corollary 2: Suppose that the input signal Ω is uniformly continuous and bounded. Let $m = p = 2$, $a_j^{(i)}, b_i \in \mathbb{S}^2$

in (7) be constant with $\Lambda_1 = \Lambda_2$, and assume b_1, b_2 non-collinear. Moreover, assume that Lemma 1 is fulfilled, and that Ω is strongly persistently exciting in the sense of Definition 2. Then, the equilibrium $(\tilde{R}, \tilde{d}) = (I_3, 0)$ is locally exponentially stable. □

The proof, reported in [22], proceeds by contradiction and shows that strong persistence of excitation of Ω is a sufficient condition to guarantee that the derivative of the integrand of (22) is not zero.

Remark 1: The scenario with two scalar measurements ($m = p = 2$) with constant vectors $a_j^{(i)}, b_i$, such that $\Lambda_1 = \Lambda_2$, requires a stronger form of persistence of excitation for the biased case than for the unbiased one. In particular, while Corollary 2 requires strong persistence of excitation of Ω, Corollary 1 can be satisfied also by constant Ω. □

Remark 2: In the case of measurements along constant inertial directions, the results of this section are consistent with the results in [2], [3], as also for scalar measurements, at least two non-collinear inertial directions are required for attitude reconstruction. When dealing with time-varying inertial directions $b_i(t)$, even a single scalar measurement ($m = 1$) can satisfy the persistent excitation conditions of Lemmas 1 and 2; for instance, a Pitot tube with sufficiently varying airspeed provides the necessary excitation [17]. □

IV. Experimental Results

This section presents experimental results to evaluate the performance of the proposed observer (8). The evaluation is based on inertial data from the BROAD dataset [23], acquired using a 9-axis Myon aktos-t IMU at a sampling rate of 286 Hz. This dataset is well-suited for benchmarking attitude estimation algorithms, as it provides accurate and well-calibrated inertial measurements from carefully executed motion sequences. The IMU was mounted on a 3D-printed rigid body with reflective markers to ensure precise ground-truth orientation tracking. The reference measurements in this setup are the gravity direction in the inertial frame $g_0 = e_3$, and the local magnetic field in the inertial frame $m_\circ$. Their corresponding body-frame measurements are provided by the accelerometer and magnetometer, respectively. We use the BROAD dataset's first three sequences, A, B, and C, that consist of undisturbed slow pure rotations. These sequences serve two complementary purposes. First, they provide an ideal benchmark to validate the theoretical results from Section III, since their rotational motions are sufficiently varied to satisfy the persistent excitation condition stated in Corollary 2, which guarantees the theoretical convergence of the proposed design even when reduced to two scalar measurements. Second, they allow us to examine how the estimation accuracy evolves as the number of scalar measurements is progressively reduced.

The dataset's recorded IMU angular rate measurements contain a residual turn-on bias of 0.17deg/s on average per axis [23]. To assess the effect of gyro bias compensation, two observer implementations are considered: (i) the attitude-only observer outlined in Section III-A, and (ii) the joint attitude-gyro bias observer (8). Both observers are evaluated under four different sets of scalar measurements obtained by selecting different combinations of accelerometer and magnetometer axes, starting from full 3-axis measurements for both sensors down to only a single axis from each sensor. The measurement sets used in each configuration are summarized in Table I.

TABLE I: Accelerometer and magnetometer components used in different measurement configurations.

Scalars	**Acc Λ_1**	**Mag Λ_2**	**Removed**
Six (all)	$[e_1\ e_2\ e_3]$	$[e_1\ e_2\ e_3]$	—
Four	$[e_2\ e_3]$	$[e_1\ e_2]$	Acc: e_1, Mag: e_3
Three	$[e_2\ e_3]$	e_2	Acc: e_1, Mag: e_1, e_3
Two	e_2	e_2	Acc: e_1, e_3, Mag: e_1, e_3

For completeness, we also report results obtained with the complementary filter [2], commonly used as a baseline in attitude estimation. However, it should be noted that a direct performance comparison with this filter is not strictly meaningful due to its fundamentally different design.

The observers were initialized with $\hat{R}(0) = I_3$ and $\hat{d}(0) = 0$, with $P(0) = 0.5 I_3$ and $V = 0.005 I_3$ for the attitude-only observer, and $P(0) = 0.5 I_6$ and $V = 0.005 I_6$ for the full attitude+gyro bias observer, and $Q = 0.05 I_m$ for both. For the complementary filter, the attitude innovation gain was set to 2.5 and the bias adaptation gain to 0.1.

A. Results and discussion

The attitude estimates obtained using the complementary filter and the proposed observer are compared against the Optitrack motion capture ground truth from the BROAD dataset. Their performance is evaluated using the root mean square error (RMSE) on the angular distance on $\mathbf{SO}(3)$, defined by $\theta := \arccos(\frac{1}{2}(\mathrm{tr}(\tilde{R}) - 1))$. The results are summarized in Table II.

Overall, the Riccati observer with full vector measurements (6 scalars) achieves the lowest total RMSE and consistently accurate estimates across all sequences, which indeed demonstrates the benefit of exploiting the complete measurement set. The complementary filter (included as a baseline) yields comparable RMSE results. Reducing the number of scalar measurements naturally leads to a slight increase in attitude errors. Still, even the two-scalar measurements observer achieves low RMSEs, demonstrating the effectiveness of the design under reduced measurement conditions under sufficiently persistently exciting motion.

An additional experiment was performed on sequence B to illustrate the convergence transients of observer (8). The attitude estimate was initialized with $\hat{R}(0)$ corresponding to a rotation of 45deg about each axis. The resulting error convergence is shown in Fig. 2. The results confirm that all observers converge to the true attitude. Convergence is fastest with full 3-axis vector measurements, followed by 4 scalars, while the 3- and 2-scalar configurations are slightly slower. This is directly related to the amount of excitation provided by the selected measurements in the conditions of Lemmas 1 and 2.

TABLE II: RMSE (in degrees) of the attitude estimation error θ for the complementary filter and the proposed observers on Sequences A, B, and C of the BROAD dataset. For each sequence, results are reported with and without bias compensation.

	Sequence A		Sequence B		Sequence C	
Observer	Attitude only	Bias compensation	Attitude only	Bias compensation	Attitude only	Bias compensation
Complementary [2]	**1.660**	**1.701**	**1.503**	1.480	**4.098**	3.885
Proposed – 6 scalars	2.723	1.903	1.829	**1.295**	4.449	**3.529**
Proposed – 4 scalars	2.677	2.087	3.499	1.770	5.634	3.621
Proposed – 3 scalars	3.037	2.399	4.347	2.552	5.443	3.563
Proposed – 2 scalars	3.502	2.835	5.279	3.242	5.543	3.665

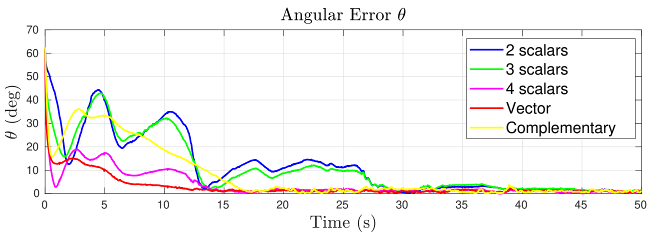

Fig. 2: Evolution of the attitude error θ for all observers with initial condition $\hat{R}(0)$ set to a $45\deg$ rotation about each axis.

V. Conclusion

We proposed a deterministic Riccati-based observer for attitude estimation on $\mathbf{SO}(3)$ using scalar measurements, explicitly accounting for gyroscope bias. The framework avoids high-dimensional embeddings and establishes persistence-of-excitation conditions linking uniform observability of the linearized dynamics to local exponential stability. A key theoretical result is that two scalar measurements under suitable excitation suffice for attitude observability, and that three are enough in the static case, matching the intrinsic dimension of $\mathbf{SO}(3)$. Experiments on the BROAD dataset confirmed these findings: the observer retained stable convergence even when reduced to two scalars, with accuracy degradation following the expected trend as measurements were progressively removed. This demonstrates the robustness and practical relevance of scalar-based estimation frameworks, especially in scenarios where full vector information is unavailable. Future work will focus on enlarging the domain of convergence of the proposed estimation scheme and on exploring lightweight constant-gain designs.

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