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On performing non-prehensile rolling manipulations: Stabilizing synchronous motions of Butterfly robots*

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Abstract— **The paper explores the challenging task of performing a non-prehensile manipulation of several balls synchronously rolling on the curved hands of Butterfly robots. Each Butterfly robot represents a standard benchmark hardware setup, comprising a DC motor rotating a butterfly-shaped frame in a vertical plane, with a ball moving freely upon it, equipped with integrated computer vision, communication, programmable control, and computation interfaces. The combined dynamics of the considered system, consisting of $N \geq 2$ such robots, is inherently underactuated, characterized by N active and N passive degrees of freedom, as well as N independent unilateral constraints that model the interactions between the frames and the balls, assuming no slipping. We focus on designing a model-based centralized feedback controller to achieve synchronized rotations of the balls. We assume the accuracy of our mathematical model and the feasibility of implementing a discretized version of the proposed continuous-time controller with a sufficiently small sampling time, that, in particular, is necessary for numerical differentiation. Relying on orbital stability of nominal periodic solution of the closed-loop system, we will experimentally check robustness to various inevitable challenges such as noises, disturbances, uncertainties, and communication delays. Hence, our concentration lies in designing an orbitally stabilizing controller for the underactuated models. The primary contribution is proposing one set of transverse coordinates, enabling transverse-linearization-based controller design, accompanied by pertinent closed-loop system analysis tools, thereby enhancing the efficacy of solving the manipulation task. Analytical and model-based arguments are validated through successful simulations and experiments conducted on two Butterfly robots, thereby emphasizing the validity and practicality of the proposed approach.**

I. MOTIVATION FOR PERFORMING ROBOTIC DYNAMIC MANIPULATIONS AND CHALLENGES

Various tasks in service and medical applications demand human-like manipulation abilities from robotic systems for automating and advancing human-involved and human-centered assignments. Meanwhile, most of the tools and environments, used by humans in every-day life, are not universal; by centuries they have been tuned and adjusted for human hands to perform specific operations, e.g. eating soup by manipulating a spoon, opening a door by pushing a handle, lightly touching an object and painting on a canvas by brushes, stitching/sewing clothes and soft materials/tissues by a needle etc. Clearly, such personalization of tools and strategies for handling external objects and environments reflects the challenge and the complexity of various actions easily performed by adults, which are often resulted from extended-in-time learning to perform various tracks as well as comprehending and advancing the skills.

At the same time, in spite of the diversity of manipulating tasks, there are a few conceptual interaction patterns defining continuous contact of a human hand and an external tool or an object or an environment along a movement: Either it is *firmly grasped* by a human hand or it *rolls* or/and *slides* on a human hand being pushed, see, e.g., [1]. The last two interaction formats, i.e. when an external object rolls or slides or both on a human hand, represent ones of the most difficult and important tasks for humans. The main difficulty comes from the necessity to overcome and/or harvest the nonlinear effects appearing due to object/environment interaction dynamics and contact conditions. A human should learn feasible and often dexterious behaviors consistent with dynamic constraints and ways to control such motions to make them repeatable and insensitive to noise and perturbations. The importance of dynamic manipulation skills is related to extended capabilities of hands and increased robustness of the human motor control system.

Developing assistive robots assumes that robotic hands will be able to perform similar grasping, manipulating, transporting, assembling, and other handling of objects as it would be done by a human. Therefore, it is attractive to analyze and explore recorded human performances for searching artifacts and incentives that can be useful in planning similar robot hand movements and in their stabilization. Meanwhile, the superficial repetition of recorded movements by a robot is unlikely to lead to success. Indeed, even small perturbations of contact conditions and differences in formats of robot's actuation substantially change the dynamical properties of the augmented system, defined by combined dynamics of the robot and the object. Such modifications preclude from the possibility for a robot hand to literally mimic and follow time references reconstructed from human motion recordings without a failure. Hence, we need alternative interpretations of human exercises both for planning similar but robot-like behaviors and for their robust control. It is worth to mention one of critical and explicit properties of a human motor control architecture: *humans have no accurate sense of time*. Therefore, developing human-like control architectures for robotic dynamic manipulation rules out most of classical

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robotics approaches based on time reference representation and time reference tracking paradigms actively used in industrial settings.

A. Time-independent representation of motions

There are alternative methods for designing model-based motion planning and motion control algorithms for robots, which are compatible with the mentioned characteristic of the human motor control system. One of them is linked to a *geometrical (nested) parametrization* of feasible forced movements of a mechanical system, when time behavior of the generalized coordinates, i.e. variables defining the current state for a mathematical model of controlled mechanical or electromechanical model, for the particular motion

$$q_1(t), \quad q_2(t), \quad \ldots, \quad q_n(t), \quad t \in [0, T]$$

of the system is re-parametrized on the same time interval as functions of a single auxiliary variable:

$$q_1(t) = \Phi_1\big(s(t)\big), \quad \ldots, \quad q_n(t) = \Phi_n\big(s(t)\big), \quad t \in [0, T].$$

Here $s(t)$ is a scalar function of time, often serving as one of generalized coordinates, conveniently chosen for the representation of a particular nominal behavior, see e.g. [2], [3], for instance, if it is expected to change monotonically as it would be for a path length along the trajectory in the configuration space or for the vertical projection of the center of mass of a walking humanoid robot.

The variable $s(t)$ is known as motion generator, see [4], while the scalar functions $\Phi_i(s)$, $i = 1, \ldots, n$, define the pattern of the synchronization among the various degrees of freedom for a model of the system along the given trajectory, called servoconstraints or virtual holonomic constraints.

The explicit time evolution of $s(t)$ might be then defined separately to meet constraints and specifications. However, for underactuated systems, the behavior of a motion generator cannot be chosen arbitrary but will be one of solutions of a scalar second order differential equation, coefficients of which are fully determined by the dynamics of the augmented system and the synchronization functions $\Phi_i(\cdot)$, $i = 1, \ldots, n$, and their derivatives, see [5].

It is importation to realize that the synchronization functions and their derivatives can be used to define new variables that vanish along the motion and can be used to characterize and to control the distance from the orbit. Using them as new coordinates allows an introduction of other important concepts for developing dynamic manipulation, which are the notions of *transverse dynamics* and *transverse coordinates*. They are well defined for a finite-time movement (like for a manipulation) and appropriate for analysis and controller designs to achieve either orbital stability for a cyclic behavior or contraction if the nominal motion is aperiodic, see [6].

The geometrical parametrization of individual trajectories for motion planning and exact or approximate stabilization of transverse dynamics have been successfully used in solving a number of challenging underactuated system examples, see [7], [8], [9], [10], [11], [12] and others. Meanwhile, the advantages of the idea for solving dynamic manipulation tasks are yet to be explored. At this point, it is worth mentioning our contribution to the topic, see [13], which allowed illustrating the successful application of the ideas for shaping stable rolling of a passive ball on a robot hand having a varying curvature, so-called Butterfly robot, see [14].

B. Key challenge and contribution

At the present study we discuss an extension of the test-bed example of [13] and propose analytical arguments for shaping synchronous non-prehensile rolling manipulations performed by several identical Butterfly robots. More specifically, despite unavoidable differences in similar hardware set-ups, such as, not matching shapes of the butterfly-shaped frames, not matching sensor noises disturbances and various dynamic parameters as well different delays and lost data in not identical transmissions, we would like to design control inputs, applied to the DC-motors of the butterfly robots, making them perform rotation of the balls not only without losing contact but also visually synchronized.

Note that the problem of creating such a motion for a single robot, solved in [13], has been an open challenge for almost 20 years, since to make the motion feasible, the velocity profile of the ball must be planned so that at each configuration the induced normal to the surface contact force simultaneously counteracts gravity, not allowing the ball to fall down, and avoids loosing the contact. This is nontrivial in particular because the curvature is varying for the particular shape, while here we add the synchronization requirement.

From the mathematical point of view, the challenge in such an assignment is primarily associated with the necessity to cope with an increased degree of underactuation of the total system, consisting of several systems with passive degrees of freedom. Note that the underactuation degree is the difference between the number of generalized coordinates and the number of the control inputs, which is equal to the number of passive degrees of freedom.

For N Butterfly robots, the number of passive degrees of freedom will be equal to N. Such systems with two and more passive degrees of freedom represent nontrivial examples.

The main contribution of the paper illustrates again the power of the original approach of [6] for orbital stabilization of movements of mechanical systems with any number of passive degrees of freedom. The analytical arguments are supported by experimental results, demonstrating robustness.

The rest of this paper is organized as follows: The preliminary information on modeling, motion planning and orbital stabilizing of non-prehensile rolling for one Butterfly robot is reviewed in Section II. The main contributions of the paper on representation of transverse dynamics and choices for transverse coordinates are collected in Section III. A discussion of the experimental platform and results on synchronization of two Butterfly robots are given in Section IV. Concluding remarks are drawn in Section V.

II. Dynamics of the Robot, Steps in Planning Perpetual Rotations and Orbital Stabilization

The Butterfly robot is one of a few benchmark examples, see [15], [16], [17], [18], [19], aimed at developing and testing non-prehensile manipulation algorithms for controlling continuous rolling of a passive ball on a hand of a robot. The setup is made of two identical figure-eight-shaped plates, rigidly placed parallel to each other with a gap in between, which is smaller than a diameter of the ball. They are actively controlled by a DC-motor, while the ball's movement is determined only by the gravity and the reaction force provided that the ball and the robot hand are in contact, see the schematic representation of the system on Fig. 1.

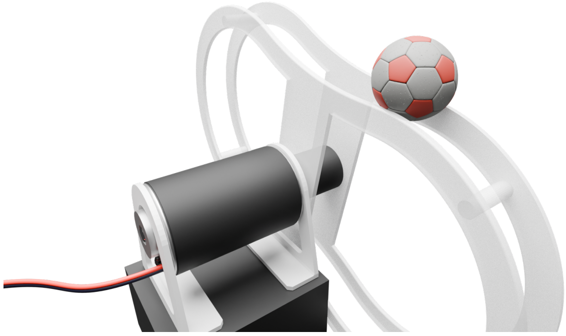

Fig. 1. A schematic view of the Butterfly robot: a ball rolling (as on rails) on boundaries of two identical rigid plates attached to a DC motor.

Under the assumptions that

1) a motion of a ball rolling on boundaries of plates can be considered as a rolling of a ball (or a cylinder),
2) for any time moment the ball and the boundary of the plates have only one point of contact and none of the bodies in contact are deformed, and
3) the ball rolls without slipping

the dynamics of the Butterfly robot can be written out using only two configuration variable:

- ϑ – the angle representing the orientation of the plates with respect to the inertia frame, countered anticlockwise from the abscissa axis, see Fig. 2,
- φ – the angle representing the angular displacement of the center of symmetry of the ball, given in the body fixed frame of the plates, countered clockwise from the ordinate axis, see Fig. 2.

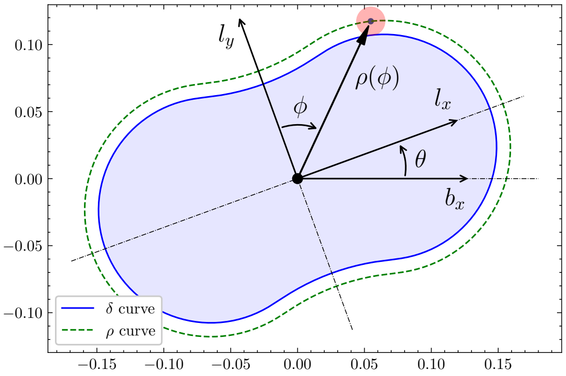

Fig. 2. Definition of two generalized coordinates ϑ and φ. The origin of the inertia frame Oxy – the vertical plane – is put in the point of actuation of the plates. By assumption, the point coincides with the center of the mass of the robot hand. The direction of the z-axis of the inertia frame is orthogonal to Oxy-plane and aimed towards the reader.

Namely, the equations of motion in the generalized coordinates $q = [\vartheta, \varphi]^T$, derived previously in [13], are

$$M(q)\,\ddot{q} + C(q,\dot{q})\,\dot{q} + G(q) = \begin{bmatrix} u \\ 0 \end{bmatrix}, \tag{1}$$

with u being a control variable representing the external torque applied by the actuator to rotate the plates. Here the 2×2 inertia matrix $M(q)$ has the form

$$M(q) = m \begin{bmatrix} \rho^2 + \frac{J_f}{m} + \frac{J_s}{m} & \left(\vec{\rho} \times \vec{\tau} - \frac{J_s}{mR}\right) s' \\ \left(\vec{\rho} \times \vec{\tau} - \frac{J_s}{mR}\right) s' & \left(1 + \frac{J_s}{mR^2}\right) s'^2 \end{bmatrix};$$

the 2×2 matrix $C(q,\dot{q})$ of Coriolis and centrifugal forces has the components

$$\begin{aligned} C_{1,1} &= m s' \vec{\tau} \cdot \vec{\rho} \dot{\varphi} \\ C_{1,2} &= m \left[s' \vec{\tau} \cdot \vec{\rho} \dot{\vartheta} + \left(s'^2 \vec{\rho} \times \vec{\kappa} + \left(\vec{\rho} \times \vec{\tau} - \tfrac{J_s}{R} \right) s'' \right) \dot{\varphi} \right] \\ C_{2,1} &= -m s' \vec{\tau} \cdot \vec{\rho} \dot{\vartheta} \\ C_{2,2} &= \left(1 + \frac{J_s}{mR^2} \right) s' s'' \dot{\varphi}; \end{aligned}$$

and the components of the vector of the generalized forces $G(q)$ due to the gravity are

$$G(q) = mg \begin{bmatrix} [0,1]^T \cdot \Pi'(\vartheta)\, \vec{\rho} \\ [0,1]^T \cdot \Pi(\vartheta)\, \vec{\tau} s' \end{bmatrix}.$$

In the expressions for $M(q)$, $C(q,\dot{q})$, $G(q)$ the following notations are used:

m	mass of the ball
J_s	moment of inertia of the ball around the z-axis
R	effective radius of the ball
g	acceleration due to gravity
J_f	moment of inertia of the plates around the z-axis
$\Pi(\vartheta)$	rotation matrix
$\vec{\rho} = \vec{\rho}(\varphi)$	radius-vector of the ball's center in $Ox'y'$ coordinate system
$\rho = \Vert\vec{\rho}\Vert$	distance between the ball's center and the origin O
$s = s(\varphi)$	natural parameter of curve $\vec{\rho}$, i.e.
$s(\varphi) = \int_0^{\varphi} \left\Vert \frac{d\vec{\rho}(\phi)}{d\phi} \right\Vert d\phi$, $s' = \frac{ds(\varphi)}{d\varphi}$, $s'' = \frac{d^2 s(\varphi)}{d\varphi^2}$	
$\vec{\tau} = \vec{\tau}(\varphi) = \frac{d\vec{\rho}}{ds}$	tangent vector to curve $\vec{\rho}(\varphi)$
$\vec{\kappa} = \vec{\kappa}(\varphi) = \frac{d^2\vec{\rho}}{ds^2}$	curvature of the curve $\vec{\rho}(\varphi)$
$\vec{a} \times \vec{b} = a_1 b_2 - a_2 b_1$	cross product of $2D$ vectors
$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$	scalar product of $2D$ vectors.

To summarize the first contribution of [13], stated in Lemma 1 and Theorem 1, one can find some of forced periodic trajectories of the Butterfly robot model (1) following the next procedure:

1) Consider a parametric family of kinematic relations

$$\vartheta = \Theta(\varphi), \tag{2}$$

with

$$\Theta(\varphi) \triangleq \varphi + \arctan\left[\frac{a \sin(2\varphi - \pi)\, [\Pi(\varphi)\,\vec{\tau}]_1 - [\Pi(\varphi)\,\vec{\tau}]_2}{a \sin(2\varphi - \pi)\, [\Pi(\varphi)\,\vec{\tau}]_2 + [\Pi(\varphi)\,\vec{\tau}]_1} \right]$$

where $[v]_i$ denotes i-th component of the vector $\vec{v}$ and a is a scalar parameter to be determined.

2) If the scalar parameter a in (2) is chosen so that the inequality

$$\Theta'(\varphi) \neq -\frac{1+\frac{J_s}{mR^2}}{\vec{\rho}(\varphi)\times\vec{\tau}(\varphi)-\frac{J_s}{mR}}\, s'(\varphi) \qquad (3)$$

holds for any $\varphi \in \mathbb{R}$, then there is a *feedforward* control input $u = u_{ff}(\varphi, \dot{\varphi})$, for the explicit formula please see [13], that ensures *the invariance* of the kinematic relation (2) and such that at least one closed-loop system solution $q = q_\star(t) = [\vartheta_\star(t), \varphi_\star(t)]^T$ has the following property:

- $\dot{\varphi}_\star(t) > 0$
- $\dot{\vartheta}_\star(t+nT) = \dot{\vartheta}_\star(t)$, $\dot{\varphi}_\star(t+nT) = \dot{\varphi}_\star(t)$
- $\vartheta_\star(t+nT) = \vartheta_\star(t) + 2\pi n$, $\varphi_\star(t+nT) = \varphi_\star(t) + 2\pi n$,

with $n \in \mathbb{Z}$ and $t \in \mathbb{R}$. Such a solution corresponds to the mathematical description of a T-periodic in velocity perpetual rolling of the ball on the robot hand. Furthermore, the behavior of $\varphi_\star(t)$ is defined as a solution of the equation

$$\alpha(\varphi)\ddot{\varphi} + \beta(\varphi)\dot{\varphi}^2 + \gamma(\varphi) = 0 \qquad (4)$$

with the coefficients given in [13], $\alpha(\varphi) \neq 0$, $\forall\, \varphi$.

The second contribution of [13] is a procedure for analytically computing transverse coordinates and transverse dynamics to the found periodic trajectory as well as arguments for designing a model-based feedback controller to achieve its orbital exponential stabilization. In particular, three transverse coordinates

$$\xi_1 := \vartheta - \Theta(\varphi), \quad \xi_2 := \dot{\vartheta} - \Theta'(\varphi)\dot{\varphi}, \quad \xi_3 := \dot{\varphi} - p(\varphi) \qquad (5)$$

were introduced, where the function $p(\varphi)$ is defined by

$$\begin{aligned} p^2(\varphi) \;\; = \;\; & \exp\left\{-2\int_{\varphi_0}^{\varphi} \frac{\beta(w)}{\alpha(w)}dw\right\} \cdot \qquad (6) \\ & \left(\dot{\varphi}_0^2 - \int_{\varphi_0}^{\varphi} \frac{2\gamma(w)}{\alpha(w)} \exp\left\{2\int_{\varphi_0}^{w} \frac{\beta(v)}{\alpha(v)}dv\right\} dw\right), \end{aligned}$$

with $\varphi_0 = \varphi_\star(t_0)$ and $\dot{\varphi}_0 = \dot{\varphi}_\star(t_0)$ chosen for an arbitrary $0 \leq t_0 \leq T$.

The analytical formulas for the linearization of quantities $\xi = [\xi_1, \xi_2, \xi_3]^T \approx [\delta\xi_1, \delta\xi_2, \delta\xi_3]^T = \delta\xi$ in the vicinity of the nominal trajectory revealed the transverse linearization

$$\tfrac{d}{d\tau}\delta\xi = A(\tau)\,\delta\xi + B(\tau)\,\delta u \qquad (7)$$

of the system dynamics, see [6]. The feedback controller, see [13], was then developed and implemented as

$$\delta u = K(\tau)\,\delta\xi \qquad (8)$$

first for stabilization of the origin of the auxiliary closed-loop dynamic system (7)-(8) and then, its modification

$$u = u_{ff}(\varphi, \dot{\varphi}) + K(\varphi)\,\xi \qquad (9)$$

with substitution $\tau = \varphi_\star(\varphi)$ was applied to the original nonlinear dynamics (1) for orbital stabilization of the found solution $q = q_\star(t)$, representing perpetual rotation.

However, applying such controllers to theoretically identical but real hardware setups results in not synchronized rotations since, even though achieved exponential orbital stabilization can be proven to be robust, even the periods of the implemented rotations would be not identical and, therefore, the motions will be visually different even if initial configurations are almost identical. This is why decentralized control strategy does not work and a redesign for the combined system of several robots is necessary.

III. Main Result: Stabilization of Rolling Movement of N Butterfly robots

Given N nominally identical copies of the Butterfly robot, their combined kinematics and dynamics are described by the next coordinates and the equations of motion:

$$q_i = [\vartheta_i, \varphi_i]^T: \quad M(q_i)\,\ddot{q}_i + C(q_i, \dot{q}_i)\,\dot{q}_i + G(q_i) = \begin{bmatrix} u_i \\ 0 \end{bmatrix} \qquad (10)$$

for $i = 1, \ldots, N$. Each of the N systems has a solution that defines periodic one-directional rolling of the ball

$$q_{1\star}(t) = [\vartheta_\star(t), \varphi_\star(t)]^T, \;\ldots,\; q_{N\star}(t) = [\vartheta_\star(t), \varphi_\star(t)]^T \qquad (11)$$

of the same period T, shaped (although not stabilized) by the corresponding feedforward controllers

$$u_i = u_{ff}(\varphi_i, \dot{\varphi}_i), \quad i = 1, \ldots, N. \qquad (12)$$

The solution (11) of each closed-loop system (10)–(12) is unstable but this individual rollings for each of the Butterfly robot can be stabilized by the corresponding modified decoupled feedback, identical to (9),

$$u_i = u_{ff}(\varphi_i, \dot{\varphi}_i) + K(\varphi_i)\begin{bmatrix} \vartheta_i - \Theta(\varphi_i) \\ \dot{\vartheta}_i - \Theta'(\varphi_i)\dot{\varphi}_i \\ \dot{\varphi}_i - p(\varphi_i) \end{bmatrix}, \quad i = 1, \ldots, N. \qquad (13)$$

However, even with everything, except for the initial conditions, for the closed-loop systems (10) with (13) being identical, due to orbital stability of the periodic solution for the models of each of the Butterfly robots, in steady-state, we will have different phase shifts in reproducing the nominal behavior, and, therefore, the rollings of all N balls will be *asynchronous*. In other words, even very small deviations in the initial conditions and in mechanical parameters as well as not identical small disturbances will lead to braking synchrony even though exponential orbital stability for the model of the closed-loop system for each individual robot leads to keeping realized trajectories in vicinities of the planned nominal orbits.

To achieve the synchrony of all the Butterfly robots, one can re-use the arguments of [6] and stabilize the associated transverse linearization of the combined (full) dynamics of $N \geq 2$ copies of the Butterfly robots (10) in a vicinity of the cycle (11). This requires introducing $4N-1$ transverse coordinates for the trajectory (11) of the combined system, i.e. scalar quantities that vanish only when all the state variables have values of the synchronized solution. Note that since the dynamics (10) has $2N$ degrees of freedom, and the dimension of its state space is doubled, i.e. equals to $4N$, the dimension of the transverse dynamics is $4N-1$ and not $3N=(4-1)N$ as would be for the isolated N systems with the decentralized individually stabilizing controllers. In other words, with the previous choice of the independent vanishing quantities we are lacking $N-1$ transverse coordinates. The next statement is the main contribution of this work, defining a set of the required functions of the system states.

Proposition 1: Consider the forced trajectory (11) of the dynamics (10), (12) of the N copies of the Butterfly robot, then the following functions

$$\left\{\begin{array}{lcl} y_i & = & \vartheta_i - \Theta(\varphi_i) \\ \dot{y}_i & = & \dot{\vartheta}_i - \Theta'(\varphi_i)\dot{\varphi}_i \end{array}\right., \quad i=1,\ldots,N \tag{14}$$

$$\left\{\begin{array}{lcl} z_j & = & \varphi_1 - \varphi_{j+1} \\ \dot{z}_j & = & \dot{\varphi}_1 - \dot{\varphi}_{j+1} \end{array}\right., \quad j=1,\ldots,N-1 \tag{15}$$

$$I = \dot{\varphi}_1 - p(\varphi_1) \tag{16}$$

with $\Theta(\varphi_i)$ and $p(\varphi_1)$ taken from (5), comprise a set of $4N-1$ transverse coordinates for the motion. ■

Proof: Each of the functions (14)–(16) is zero on the nominal trajectory and can be taken as one of transverse coordinates. Hence, it is sufficient to show that these functions are independent in a vicinity of the nominal trajectory. To this end, let us consider the Jacobian of the transformation

$$[q_1;\ldots;q_N;\,\dot{q}_1;\ldots;\dot{q}_N] \to [y;\,z;\,\dot{y};\,\dot{z};\,I]$$

with

$$\begin{array}{lcllcl} y & = & [y_1,\ldots,y_N]^T, & \dot{y} & = & [\dot{y}_1,\ldots,\dot{y}_N]^T \\ z & = & [z_1,\ldots,z_{N-1}]^T, & \dot{z} & = & [\dot{z}_1,\ldots,\dot{z}_{N-1}]^T \end{array}$$

and check its rank in a vicinity of the motion. The Jacobian $J\big(q;\,\dot{q}\big)$ is the $(4N-1)\times 4N$-matrix function given by

$$\begin{bmatrix} \delta y \\ \delta z \\ \delta\dot{y} \\ \delta\dot{z} \\ \delta I \end{bmatrix} = J(q;\dot{q})\begin{bmatrix} \delta q \\ \delta\dot{q} \end{bmatrix}, \quad J(q;\dot{q}) = \begin{bmatrix} \partial y/\partial q & \partial y/\partial\dot{q} \\ \partial z/\partial q & \partial z/\partial\dot{q} \\ \partial\dot{y}/\partial q & \partial\dot{y}/\partial\dot{q} \\ \partial\dot{z}/\partial q & \partial\dot{z}/\partial\dot{q} \\ \partial I/\partial q & \partial I/\partial\dot{q} \end{bmatrix} \tag{17}$$

Taking into account (14)–(16), we have

$$\delta y = \begin{bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \\ \vdots \\ \delta\vartheta_N \end{bmatrix} + \begin{bmatrix} \Theta'_1(\varphi_1) & 0 & \ldots & 0 \\ 0 & \Theta'_1(\varphi_2) & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & \Theta'_1(\varphi_N) \end{bmatrix} \begin{bmatrix} \delta\varphi_1 \\ \delta\varphi_2 \\ \vdots \\ \delta\varphi_N \end{bmatrix}$$

$$\delta z = \begin{bmatrix} 1 & -1 & 0 & \ldots & 0 \\ 1 & 0 & -1 & \ldots & 0 \\ \vdots & & & \ddots & \vdots \\ 1 & 0 & \ldots & 0 & -1 \end{bmatrix} \begin{bmatrix} \delta\varphi_1 \\ \delta\varphi_2 \\ \vdots \\ \delta\varphi_N \end{bmatrix}$$

Let us denote the matrix expressions above as

$$\begin{bmatrix} \delta y \\ \delta z \end{bmatrix} = \Lambda_1(\varphi_{1\ldots N}) \begin{bmatrix} \delta\vartheta_{1\ldots N} \\ \delta\varphi_{1\ldots N} \end{bmatrix},$$

so that $\Lambda_1(\varphi_{1\ldots N})$ becomes a low-triangular matrix function with constant blocks on the diagonal. Then we can compactly express other components of the Jacobian as follows

$$\begin{bmatrix} \delta\dot{y} \\ \delta\dot{z} \end{bmatrix} = \left[\begin{array}{c|c} 0_{N\times N} & \Lambda_2(\varphi_{1\ldots N},\dot{\varphi}_{1\ldots N}) \\ \hline 0_{N\times N} & 0_{N\times N} \end{array}\right] \begin{bmatrix} \delta\vartheta_{1\ldots N} \\ \delta\varphi_{1\ldots N} \end{bmatrix} + \Lambda_1(\varphi_{1\ldots N}) \begin{bmatrix} \delta\dot{\vartheta}_{1\ldots N} \\ \delta\dot{\varphi}_{1\ldots N} \end{bmatrix}$$

with

$$\Lambda_2(\cdot) = \begin{bmatrix} \Theta''(\varphi_1)\dot{\varphi}_1 & 0 & \ldots & 0 \\ 0 & \Theta''(\varphi_2)\dot{\varphi}_2 & \ldots & 0 \\ \vdots & & \ddots & \\ 0 & \ldots & 0 & \Theta''(\varphi_N)\dot{\varphi}_N \end{bmatrix}$$

Obviously, the rank of the matrix $\Lambda_1(\varphi_{1\ldots N})$ is constant and equal to $2N-1$. Therefore, irrespective of the choice of $\Theta(\varphi_i)$, the rank of the Jacobian $J\big(q;\,\dot{q}\big)$ has a lower bound equal to $2(2N-1)=4N-2$. To finish the proof, one needs to observe that the component

$$\delta I = \delta\dot{\varphi}_1 - p'(\varphi_1)\cdot\delta\varphi_1$$

is independent of the variations of y, z and their derivatives ensuring that the rank of the Jacobian (17) is indeed equal to $4N-1$. ■

With the set of transverse coordinates (14)-(16) the analytical arguments of [6], combined with the associated numerical methods, see [20], [21], developed for designing controllers for orbital stabilization of cycles of underactuated mechanical systems can be directly applied for synchronizing non-prehensile manipulations, defined by the solution (11) of the combined model of N Butterfly robots (10).

Meanwhile, this set of the variables and the corresponding transverse linearization can be also used for analysis and verification of different (even *ad hoc*) feedback designs. To this end, one can suggest various modifications of the controllers (13) by adding terms interconnecting different subsystems, for instance, by the couplings present in Toda lattices, see [22], [23]

$$\begin{array}{rcl} u_i & = & \bar{u}_i + \sum\limits_{j\neq i} k_{ij}(\varphi_i - \varphi_j) \\ & = & \bar{u}_i + \sum\limits_{j\neq i} k_{ij}\left\{z_j - z_i\right\}, \quad i=1,\ldots,N \end{array} \tag{18}$$

with the defined as in (13)

$$\bar{u}_i = u_{ff}(\varphi_i,\dot{\varphi}_i) + K(\varphi_i)\begin{bmatrix} \vartheta_i - \Theta(\varphi_i) \\ \dot{\vartheta}_i - \Theta'(\varphi_i)\dot{\varphi}_i \\ \dot{\varphi}_i - p(\varphi_i) \end{bmatrix}, \quad i=1,\ldots,N. \tag{19}$$

Above, $\{k_{ij}\}$ are constant parameters to be defined.

However, it can be verified that the orbital exponential stability of the cycle (11) for the closed-loop system (10), (18) is not guaranteed for small values of coupling coefficients $\{k_{ij}\}$. Indeed, whenever $\{k_{ij}\}$ are zeros, then the exponential orbital stability of cycles achieved by individual controllers for each of Butterfly robots ensures only that the components of the transverse coordinates z and $\dot{z}$ – describing synchronization in-between the systems – will behave as follows: z will be bounded and $\dot{z}$ will converge to zero. This implies, at best, the *marginal stability* of the transverse linearization of the augmented dynamics (10), (18) without couplings. Meanwhile, it can be potentially unstable when arbitrarily small coupling terms appear in the feedback controller (18). Hence, another design is needed and can be performed using the following statement.

Lemma 1: Consider the closed-loop system (10), (18) and the associated linearization of the transverse coordinates

$$x_\perp = [y; \dot{y}; z; \dot{z}; I]$$

defined by the relations (14)-(16) in a vicinity of (11)

$$\frac{d}{d\tau}\delta x_\perp = A_{cl}(\tau, k_{ij})\,\delta x_\perp. \tag{20}$$

If the origin of the linear periodic system (20) is asymptotically stable, then the cycle (11) of (10), (18) is orbitally exponentially stable. ■

Proof follows from the standard arguments on properties of the transverse linearization, see [24], [25], [26], [27], [28], [29], [30], while the explicit expressions for $A_{cl}(\tau, k_{ij})$ can be computed as in our previous works.

As well-known, see e.g. [31], the origin of (20) is exponentially stable if and only if all $4N-1$ eigenvalues of the monodromy matrix are strictly inside the unit circle in the complex plane. It can be shown that when all $k_{ij} = 0$, $3N$ eigenvalues are strictly inside the unit circle, while the other $N-1$ are on the border, resulting in the lack of synchronization described above. It may be possible to find the coefficients using numerical search to fix this problem while we leave the theoretical investigation on when this is possible for the future study.

IV. EXPERIMENTAL RESULTS

Let us describe results of our experimental studies in synchronizing of non-prehensile manipulations performed on two Butterfly robots, each of which has an open-source software interface and the following components: the computer vision system based on the Basler USB3 camera with 160 fps and 1.3 MPx frame resolution; the Maxon RE50 DC motor coupled with the SCANCON encoder of 8192 ppr; the distributed computing system consisting of a PC Intel NUC and BeagleBone Black used for image processing and real-time control, correspondingly. The plates are made of transparent polycarbonate, see the robot on Fig. 3.

The physical parameters of the system (masses, inertia, dimensions, shape of the robot hand etc.) as well as mathematical models for the system dynamics were provided by the producer. The model-based motion planning for a non-prehensile rolling of the ball on the robot hand was done as in [13], based on the parametric relations (2) and on consecutive analysis of the dynamics of motion generators (4). The decoupled identical controllers for shaping stable rollings for each of the Butterfly robots were developed as in [13] through stabilization of the corresponding transverse linearization, see Eqns. (7)-(9). For synchronizing non-prehensile rollings of two Butterfly robots we have followed the procedure of [6] and developed several orbitally stabilizing controllers taking advantage of transverse linearization of the full set of $4N-1$ transverse coordinates (14)-(16).

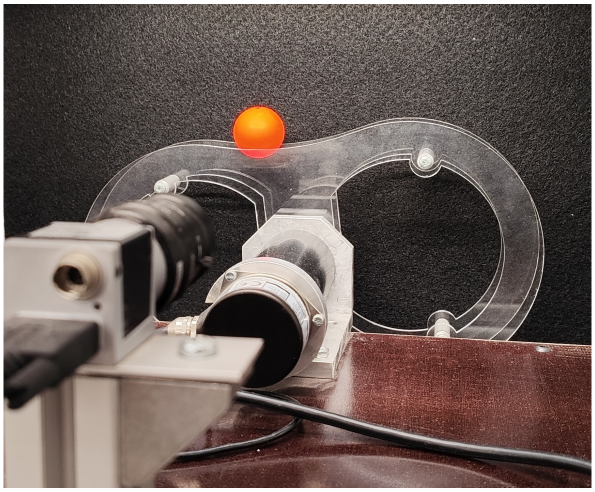

Fig. 3. The Butterfly robot, produced by RobotikUm AB, consisting of the frame, computer vision and control systems.

Furthermore, to illustrate the alternative method we have tested a low complexity solution (18) for an orbitally stabilizing feedback controller design. Namely, the systems have been governed by the original decoupled feedback laws, while one of controllers was modified by adding the asymmetric interconnection between two Butterfly robots

$$u_1 = \bar{u}_1, \qquad u_2 = \bar{u}_2 + k_{12}\,(\varphi_1 - \varphi_2),$$

where $\bar{u}_1$ and $\bar{u}_2$ are the individual controllers (19) for the first and for the second Butterfly robots accordingly, designed as in [13].

Since $N = 2$, the dimension of the transverse dynamics is equal to $4N-1 = 7$ while we know that with $k_{12} = 0$, $3N = 6$ characteristic multipliers, which are the eigenvalues of the monodromy matrix, are strictly inside the unit circle while $1 = 7-6$ of them is equal to 1.

Using analytical expressions and standard numerical computations, one can find the range of values for the coefficient k_{12} when absolute values of all 7 eigenvalues are less than 1, that corresponds to an orbital exponential stability of the cycle for the system of two Butterfly robots. For instance, if $k_{12} = 0.125$, then the amplitudes of the characteristic multipliers were found to be real and all less than $0.622 < 1$. Following Lemma 1, two Butterfly robots are synchronized. The reader is invited to check the recordings of the experiment in the attached movie and Fig. 4 below.

V. DISCUSSION AND CONCLUDING REMARKS

This work illustrates the applicability of the general approach of [6] for controlling dynamic manipulation, provided that a feasible behavior of the system (possibly unstable) is found. The method is model-based and assumes the knowledge of contact models and dynamics of an external object or environment to be manipulated.

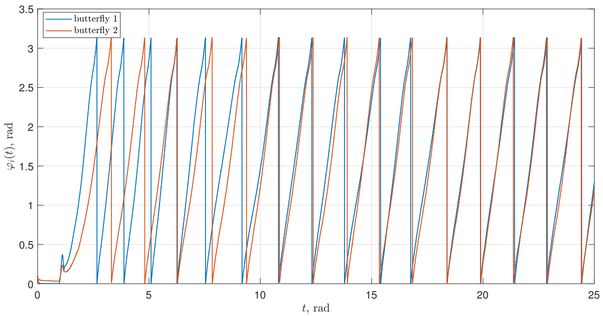

Fig. 4. Time evolution of recorded signals for the rotation angles of the two Butterfly robots, $\varphi_1(t) \mod 2\pi$ and $\varphi_2(t) \mod 2\pi$. One can see that the periodic motions are gradually synchronized, i.e. that $|\varphi_1 - \varphi_2| \to 0$.

The technical contribution of the paper is focused on developing a feedback strategy for controlling synchronous non-prehensile rolling of passive balls on hands of several Butterfly robots. It is achieved through an analytical description of transverse coordinates for the motion. These coordinates are not uniquely defined for the cycle of the augmented system dynamics and the reader can easily provide new ones repeating the arguments of Proposition 1 for establishing the independence of an alternative candidates for characterizing the transverse dynamics. In the case study, one can consider and explore independence of z-variables defined as

$$z_i = \varphi_2 - \varphi_i \quad \text{or} \quad z_i = \sin(\varphi_1 - \varphi_i), \quad i = 2, 3, 4, \ldots, N$$

or any other smooth functions equal to zero on the nominal motion. The same argument is valid for defining $\dot{z}$-components, which are not necessary to be chosen as the corresponding derivatives of z. There are many alternative choices, as discussed, in particular, in [32].

Another contributions of the paper are in efforts for organizing the range of experiments and performing synchronous dynamic manipulation on several robotic platforms. The success of the experimental study fully supports the model-based approach elaborated in the text and emphasizes the scalability of the method. It gives a hope for solving other challenging dynamic manipulation assignments.

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