2025 IEEE/SICE International Symposium on System Integration (SII)
January 21-24, 2025, Munich, Germany

Unknown Input Observer for Takagi-Sugeno Fuzzy Bilinear System with Input and Output Disturbances

Jun Yoneyama[1]

Abstract— **An estimation of the state variables for systems with disturbances is an important problem. In a practical situation, not all the state variables are measurable, and disturbance noises come into the system. Especially, it is difficult to estimate the state variables of complicated nonlinear systems with input and output disturbances. In this paper, observer design methods for a discrete-time Takagi-Sugeno fuzzy bilinear system with unknown inputs and input/output disturbances are proposed. Our observer filters out unknown input and estimates the state and input/output disturbances of the system. Since Takagi-Sugeno fuzzy bilinear system represents a quite large class of nonlinear systems, an unknown input observer that estimates the state and input/output disturbance of Takagi-Sugeno fuzzy bilinear system with unknown inputs and disturbances is essential in many engineering fields. Observer design has started with a parallel distributed observer(PDO), which is constructed with local linear observers and the appropriate grade of the membership functions. However, design conditions for PDO are very conservative. To overcome such disadvantage, non-PDO with multiple Lyapunov matrices technique is proposed to design our observer in this paper. Such a design method is based on a multiple Lyapunov function with a sum of the membership functions. This method drastically reduces the conservatism. To demonstrate the validity of our proposed observer design approach, an illustrative numerical example is provided. Lastly, we end with concluding remarks.**

I. INTRODUCTION

An observer design problem is important issues in system control and estimation. As for a stabilizing controller design, exact estimation of the system state variables from the measured output and the known input is essential. Especially when a system has unknown external inputs, estimation of the exact value of the state variables and getting rid of unknown disturbances are a difficult problem but they are indispensable in control design. Hence, quite a few works on observer design have been carried out. However, much of them have focused on linear observer design and there is still a room for observer design of nonlinear systems. One of the approaches of observer design for nonlinear systems is to design an observer for a Takagi-Sugeno fuzzy system, which is well-known to represents a large class of nonlinear systems(see [3], [5], [19], [20], [21], [26] for example). Takagi-Sugeno fuzzy system can describe a general nonlinear system and even fuzzy bilinear system can have potential to be a better representation for it([11], [13], [14], [18], [30], [31]). In fact, a fuzzy bilinear system generally has a less number of subsystems than the standard fuzzy system. Due to this characteristics of a fuzzy bilinear system, it can drastically reduces the conservatism in stability and control design conditions. Consequently, resulting observer design conditions for fuzzy bilinear systems become so much less conservative(see [32], [33].) than those of the standard fuzzy systems.

Research works on observer design for standard fuzzy systems have been published in [2], [4], [5], [15], [16], [28], [29]. An observer design for fuzzy bilinear systems has also been proposed in [11], [13], [14], and [23], some of which take care of the observer design that removes unknown external disturbances, which is called an unknown input observer(UIO). It does not only estimates the state variables, but also gets rid of unknown inputs. A UIO has a more practical estimation mechanism for actual systems with unknown input signals. One of the conventional and typical observers for a fuzzy system is a parallel distributed observer(PDO). A PDO is composed with local linear observers, which correspond to local linear subsystems in the standard Takagi-Sugeno fuzzy system. It is well-known, however, that a PDO with common Lyapunov matrix approach has conservative observer design conditions because they are based on a single Lyapunov function. Then, it was generalized to a PDO design with multiple Lyapunov matrix technique and validity of PDO became wider. For fuzzy bilinear systems, a PDO with single Lyapunov matrix approach is still commonly used, and hence generalization on the observer design is somehow necessary. In [1], [8], and [27], a non-PDO design and an observer-based parallel distributed compensator(PDC) for fuzzy systems were proposed and these references attempted to reduce conservatism, but in their design methods, still traditional common Lyapunov matrix approach that provides conservative design conditions, was employed. Most of these results are on a continuous-time Takagi-Sugeno fuzzy system, and only a few observer design methods for discrete-time counterpart have been proposed([9], [25], [34], [35]). In addition, discrete-time systems are important for practical use of digital devices.

In this paper, we attempt to estimate disturbances which come into the input as well as unknown external inputs in the system. In other words, we consider a UIO design with estimation of input and disturbances(UIOED) for discrete-time version of Takagi-Sugeno fuzzy bilinear systems in this paper. Concerned with conservative approach to a UIO, we adopt a new class of fuzzy observers, called non-PDO, and attempt to derive improved observer design conditions by taking into account the stability of the error between the actual state variables and their estimates. For non-PDO, we employ a fuzzy Lyapunov function, which has a polytopic

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[1] Jun Yoneyama is with Department of Electrical Engineering and Electronics, Aoyama Gakuin University, Sagamihara, Japan `yoneyama@ee.aoyama.ac.jp`

function with the membership functions. Moreover, we take care of the general case of higher-order input and output disturbances, which can be described by a discrete-time system. Then, we propose UIOED design methods for such a nonlinear system with unknown inputs and disturbances coming into input. After a simple numerical example demonstrates our observer design methods to prove the validity of our approach, we will give concluding remarks,

For simplicity, the following notation and similar one will be used:

$$P_\lambda = \sum_{i=1}^{r} \lambda_i(\xi_k) P_i,\ P_{\lambda^-} = \sum_{i=1}^{r} \lambda_i(\xi_{k-1}) P_i$$

$$P_{\lambda\lambda} = \sum_{i=1}^{r}\sum_{j=1}^{r} \lambda_i(\xi_k)\lambda_j(\xi_k) P_{ij}$$

II. DISCRETE-TIME FUZZY BILINEAR SYSTEMS

First of all, we introduce a class of discrete-time Takagi-Sugenso fuzzy bilinear systems with unknown inputs and disturbances under consideration.

The discrete-time Takagi-Sugeno(T-S) fuzzy bilinear model with the fuzzy if-then rules is assumed to be given by:

$$\begin{array}{ll} \textbf{IF} & \xi_1 \text{ is } M_{i1} \text{ and } \cdots \text{ and } \xi_p \text{ is } M_{ip} \\ \textbf{THEN} & x_{k+1} = A_i x_k + B_i(u_k - d_{2k}) + M_i y_k u_k \\ & \quad + N_i x_k u_k + F_i d_{1k},\ i = 1, \cdots, r \\ & y_k = C(x_k - d_{3k}) \end{array}$$

where $\xi_i,\ i = 1, \cdots, p$ and M_{ij} are the premise variables and fuzzy sets, respectively. $x_k \in \Re^n$, $u_k \in \Re$, $d_{1k} \in \Re^{m_1}$, $d_{2k} \in \Re^{m_2}$, $d_{3k} \in \Re^{m3}$ and $y_k \in \Re^q$ are the state variable, the control input, the unknown external input, the input and output disturbances, and the measurement output, respectively. A_i, B_i, N_i, F_i and C denote constant matrices of appropriate dimensions, and matrix C is assumed to be of column full rank. For simple notation, we write $\xi = [\xi_1 \quad \cdots \quad \xi_p]^T$. The assumption on the premise variable $\xi_i,\ i = 1, \ldots, p$ is that they are measurable.

The dicrete-time fuzzy bilinear system is described by

$$\begin{aligned} x_{k+1} &= \sum_{i=1}^{r} \lambda_i(\xi)\{A_i x_k + B_i(u_k - d_{2k}) + M_i y_k u_k \\ &\qquad + N_i x_k u_k + F_i d_k\} \\ &\overset{def}{=} A_\lambda x_k + B_\lambda(u_k - d_{2k}) + M_\lambda y_k u_k + N_\lambda x_k u_k \\ &\quad + F_\lambda d_{1k} \qquad (1) \\ y_k &= C(x_k - d_{3k}) \qquad (2) \end{aligned}$$

respectively, where

$$\lambda_i(\xi) = \frac{\beta_i(\xi)}{\sum_{k=1}^{r} \beta_k(\xi)},\quad \beta_i(\xi) = \prod_{j=1}^{p} M_{ij}(\xi_j) \qquad (3)$$

and $M_{ij}(\cdot)$ is the grade of the membership function of M_{ij}. It is assumed that

$$\beta_i(\xi_k) \geq 0,\ i = 1, \cdots, r,\ \sum_{i=1}^{r} \beta_i(\xi_k) > 0$$

for all ξ_k. Consequently, $\lambda_i(\xi_k)$ satisfies

$$\lambda_i(\xi_k) \geq 0,\ i = 1, \cdots, r,\ \sum_{i=1}^{r} \lambda_i(\xi_k) = 1$$

for all ξ_k.

The input and output disturbance d_{2k} concerned in (1) are assumed to be unknown but known to be constant. In other words, d_{2k} and d_{3k} follow

$$\begin{aligned} d_{2k+1} &= d_{2k} \qquad (4) \\ d_{3k+1} &= d_{3k} \qquad (5) \end{aligned}$$

Combining (1), (2) and (4), we have

$$\begin{aligned} \bar{x}_{k+1} &= \bar{A}_\lambda \bar{x}_k + \bar{B}_\lambda u_k + \bar{M}_\lambda y_k u_k + N_\lambda \bar{x}_k u_k \\ &\quad + \bar{F}_\lambda d_{1k} \qquad (6) \\ y_k &= \bar{C}\bar{x}_k \qquad (7) \end{aligned}$$

where $\bar{x}_k = [x_k^T \quad d_{2k}^T \quad d_{3k}^T]^T$ and

$$\bar{A}_\lambda = \begin{bmatrix} A_\lambda & -B_\lambda & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix},\ \bar{B}_\lambda = \begin{bmatrix} B_\lambda \\ 0 \\ 0 \end{bmatrix},\ \bar{M}_\lambda = \begin{bmatrix} M_\lambda \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{N}_\lambda = \begin{bmatrix} N_\lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \bar{F}_\lambda = \begin{bmatrix} F_\lambda \\ 0 \\ 0 \end{bmatrix},\ \bar{C} = [C \quad 0 \quad -I]$$

Remark 2.1: It is assumed in this paper that the output equation (2) is linear. For the case of the nonlinear output equation, see [33] and [35] for the detail. Those references discuss the nonlinear output and the techniques there can be applied to our system (6) and (7). We also make an assumption on the measurable premise variables, but the case of the unmeasurable premise variable is discussed in [33] and [35].

Our problem is to construct an observer that gets rid of unknown inputs in the system and estimates the state variables and the input disturbance and the output disturbance in (6) and (7), respectively. Such an observer is said to be an unknown input observer with estimation of the disturbance(UIOED).

Before we complete this section, useful lemmas are introduced for our main results in the later section.

Lemma 2.1: ([24])

$$\sum_{i=1}^{r}\sum_{j=1}^{r} \lambda_i(\xi_k)\lambda_j(\xi_k)\Phi_{ij} < 0$$

holds if the following is satisfied.

$$\begin{aligned} \Phi_{ii} < 0, &\qquad i = 1, \cdots, r \\ \frac{2}{r-1}\Phi_{ii} + \Phi_{ij} + \Phi_{ji} < 0, &\quad i \neq j,\ i, j = 1, \cdots, r \end{aligned}$$

Lemma 2.2: ([17]) Consider $x \in \Re^n$ and $Q = Q^T \in \Re^{n\times n}$ and $R \in \Re^{n\times n}$ such that $\text{rank}(R) < n$. The followings are equivalent:

1) $x^T Q x < 0$ for all $x \in \Re^n$ such that $x \neq 0, Rx = 0$.
2) There exists $M \in \Re^{n\times m}$ such that $Q + MR + R^T M^T < 0$.

III. FUZZY OBSERVER DESIGN

This section will proposes design methods for the observer problem formulated in Section II. Based on asymptotic stability conditions of the error system between the actual states and their estimates via fuzzy Lyapunov function, we will try to derive less conservative conditions of a UIOED for the fuzzy bilinear system (1) and (2). Under such design conditions, we will make a proposal design methods of the UIOED.

A. New Non-PDO Design

When it comes to an observer design of fuzzy system, the parallel distributed observer(PDO) is the standard one. However, design conditions for PDO, are generally conservative and can not allow to design an observer for a large class of fuzzy systems. To avoid such a disadvantage, we propose a new non-parallel distributed observer(non-PDO):

$$\begin{aligned} z_{k+1} &= \left(\sum_{i=1}^{r} \lambda_i(\xi_k)\hat{G}_i\right)^{-1} \sum_{i=1}^{r}\sum_{j=1}^{r} \lambda_i(\xi_k)\lambda_j(\xi_k) \\ &\quad \times(\hat{A}_{ij}z_k + \hat{B}_{ij}u_k + \hat{M}_{ij}y_ku_k + \hat{L}_{ij}y_k) \\ &= \hat{G}_\lambda^{-1}(\hat{A}_{\lambda\lambda}z_k + \hat{B}_{\lambda\lambda}u_k + \hat{M}_{\lambda\lambda}y_ku_k \\ &\quad +\hat{L}_{\lambda\lambda}y_k) \qquad (8) \\ \hat{x}_k &= z_k - \hat{H}y_k \qquad (9) \end{aligned}$$

where $z_k \in \Re^n$ is the vector and $\hat{x}_k \in \Re^n$ is the estimate of $\bar{x}_k$. $\hat{A}_{ij}$, $\hat{B}_{ij}$, $\hat{M}_{ij}$, $\hat{G}_i$, $\hat{L}_{ij}$ and $\hat{H}$ are matrices to be calculated.

We are now ready to propose the first main results.

Theorem 3.1: A UIOED of the fuzzy bilinear system (1) and (2) is designed by (8) and (9) if there exist matrices $P_{ij} > 0,\ \hat{G}_i,\ \hat{K}_{ij},\ i,j = 1,\cdots,r$ and T such that

$$\Phi_{ii}^{l} < 0,\ i,l = 1,\cdots,r \qquad (10)$$

$$\frac{2}{r-1}\Phi_{ii}^{l} + \Phi_{ij}^{l} + \Phi_{ji}^{l} < 0,\ i,j,l = 1,\cdots,r,\ i \neq j \qquad (11)$$

$$\Phi_{ij}^{l} = \begin{bmatrix} -P_l & * \\ \hat{G}_jT\bar{A}_i - \hat{K}_{ij}\bar{C} & -\hat{G}_j - \hat{G}_j^T + P_i \end{bmatrix} \qquad (12)$$

$$T\bar{F}_i = 0,\ i = 1,\cdots,r \qquad (13)$$

Observer matrices are calculated as

$$\begin{aligned} \hat{A}_{ij} &= \hat{G}_iT\bar{A}_i - \hat{K}_{ij}\bar{C},\ i,j = 1\cdots,r \qquad (14) \\ \hat{H} &= (T-I)\bar{C}^{+} \qquad (15) \\ \hat{L}_{ij} &= \hat{K}_{ij} - \hat{A}_{ij}\hat{H},\ i,j = 1\cdots,r \qquad (16) \\ \hat{B}_{ij} &= \hat{G}_iT\bar{B}_i,\ i,j = 1\cdots,r \qquad (17) \\ \hat{M}_{ij} &= \hat{G}_iT(\bar{M}_i\bar{C} + \bar{N}_i)\bar{C}^{+},\ i,j = 1\cdots,r \qquad (18) \end{aligned}$$

where $\bar{C}^{+}$ denotes a pseudo-inverse matrix of $\bar{C}$.

Proof: To analyze the existence of an observer, we need to evaluate the error e_k between the state x_k and the estimate $\hat{x}_k$;

$$e_k = \hat{x}_k - \bar{x}_k = z_k - T\bar{x}_k$$

where

$$T = I + \hat{H}\bar{C} \qquad (19)$$

The error e_k satisfies

$$\begin{aligned} e_{k+1} &= z_{k+1} - T\bar{x}_{k+1} \\ &= \hat{G}_\lambda^{-1}(\hat{A}_{\lambda\lambda}z_k + \hat{B}_{\lambda\lambda}u_k + \hat{M}_{\lambda\lambda}y_ku_k \\ &\quad +\hat{L}_{\lambda\lambda}y_k) - T(\bar{A}_\lambda\bar{x}_k + \bar{B}_\lambda u_k \\ &\quad +\bar{M}_\lambda y_ku_k + N_\lambda\bar{x}_ku_k + \bar{F}_\lambda d_{1k}) \\ &= \hat{G}_\lambda^{-1}\hat{A}_{\lambda\lambda}e_k + (\hat{G}_\lambda^{-1}\hat{A}_{\lambda\lambda}T + \hat{G}_{\lambda\lambda}^{-1}\hat{L}_{\lambda\lambda}\bar{C} \\ &\quad -T\bar{A}_\lambda)\bar{x}_k + (\hat{G}_\lambda^{-1}\hat{B}_{\lambda\lambda} - T\bar{B}_\lambda)u_k \\ &\quad +(\hat{G}_\lambda^{-1}\hat{M}_{\lambda\lambda}\bar{C} - T\bar{M}_\lambda\bar{C} - T\bar{N}_\lambda)\bar{x}_ku_k \\ &\quad -T\bar{F}_\lambda d_{1k} \qquad (20) \end{aligned}$$

If the following equations are satisfied:

$$\hat{G}_\lambda^{-1}\hat{A}_{\lambda\lambda}T + \hat{G}_\lambda^{-1}\hat{L}_{\lambda\lambda}\bar{C} - T\bar{A}_\lambda = 0 \qquad (21)$$

$$\hat{G}_\lambda^{-1}\hat{B}_{\lambda\lambda} - T\bar{B}_\lambda = 0 \qquad (22)$$

$$\hat{G}_\lambda^{-1}\hat{M}_{\lambda\lambda}\bar{C} - T(\bar{M}_\lambda\bar{C} + \bar{N}_\lambda) = 0 \qquad (23)$$

$$T\bar{F}_\lambda = 0 \qquad (24)$$

it follows from (20) that the error becomes an equation:

$$e_{k+1} = \hat{G}_\lambda^{-1}\hat{A}_{\lambda\lambda}e_k \qquad (25)$$

If the error system (25) is asymptotically stable, e_k goes to zero as k goes to infinity. In other words, equations (8) and (9) become an observer for the system (6) and (7).

Next, we will try to obtain conditions for (25) to be asymptotically stable. To this end, we consider the following fuzzy Lypunov function as a Lyapunov function candidate:

$$V(e_k) = e_k^T P_{\lambda^-} e_k \qquad (26)$$

where $P_i > 0,\ i = 1,\cdots,r$ in P_{λ^-} to be obtained. To determine stability conditions for (25), we analyze the difference Δ of $V(e_k)$:

$$\begin{aligned} \Delta &= V(e_{k+1}) - V(e_k) \\ &= [e_k \quad e_{k+1}] \begin{bmatrix} -P_{\lambda^-} & 0 \\ 0 & P_\lambda \end{bmatrix} \begin{bmatrix} e_k \\ e_{k+1} \end{bmatrix} < 0 \qquad (27) \end{aligned}$$

(25) is written as

$$[\hat{G}_\lambda^{-1}\hat{A}_{\lambda\lambda} \quad -I] \begin{bmatrix} e_k \\ e_{k+1} \end{bmatrix} = 0 \qquad (28)$$

Then, applying Lemma 2.2 to (27) and (28) with $M = \begin{bmatrix} 0 \\ \hat{G}_\lambda \end{bmatrix}$, we have

$$\begin{bmatrix} -P_{\lambda^-} & \hat{A}_{\lambda\lambda}^T \\ \hat{A}_{\lambda\lambda} & -\hat{G}_\lambda - \hat{G}_\lambda^T + P_\lambda \end{bmatrix} < 0 \qquad (29)$$

It follows from (19) and (21) that

$$\begin{aligned} \hat{A}_{\lambda\lambda} &= \hat{A}_{\lambda\lambda}T - \hat{A}_{\lambda\lambda}\hat{H}\bar{C} \\ &= \hat{G}_\lambda T\bar{A}_\lambda - (\hat{L}_{\lambda\lambda} + \hat{A}_{\lambda\lambda}\hat{H})\bar{C} \\ &= \hat{G}_\lambda T\bar{A}_\lambda - \hat{K}_{\lambda\lambda}\bar{C} \qquad (30) \end{aligned}$$

where

$$\hat{K}_{\lambda\lambda} = \hat{L}_{\lambda\lambda} + \hat{A}_{\lambda\lambda}\hat{H} \qquad (31)$$

Substituting (30) into (29), we find

$$\begin{bmatrix} -P_{\lambda^-} & * \\ \hat{G}_\lambda T\bar{A}_\lambda - \hat{K}_{\lambda\lambda}\bar{C} & -\hat{G}_\lambda - \hat{G}_\lambda^T + P_\lambda \end{bmatrix} < 0 \qquad (32)$$

makes the system (25) asymptotically stable. It is easy to rewrite (32) as

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{l=1}^{r}\lambda_i(\xi_k)\lambda_j(\xi_k)\lambda_l(\xi_{k-1}) \times \begin{bmatrix} -P_l & * \\ \hat{G}_j T\bar{A}_i - \hat{K}_{ij}\bar{C} & -\hat{G}_j - \hat{G}_j^T + P_i \end{bmatrix} < 0$$

Applying Lemma 2.1, we have (10)-(11). Observer matrices (14)-(18) readily follow from (30), (19), (31), (22) and (23), respectively.

Remark 3.1: Due to equality constrain of T in (13), conditions (10)-(11) are not completely LMIs. In order to solve matrix variables in (10)-(11), we may follow the algorithm:
1. First, find T satisfying (13).
2. Substitute such a T in (10)-(11), which become LMIs.
3. Solve LMIs (10)-(11) for P_j, $\hat{G}_j$ and $K_{ij}, i,j = 1,\cdots,r$.
4. Calculate observer matrices (14)-(18).

Remark 3.2: Theorem 3.1 is an extension of the previous results because the fuzzy Lyapunov function (26) is adopted. On the other hand, the single Lyapunov function was employed to obtain the previous results([34]). In fact, observer described by (8) and (9) becomes to the PDO if $P_i = P$, $\forall\ i$.

B. PDO Design

For many years, a parallel distributed observer(PDO) has been adopted for fuzzy systems. This is because design conditions of PDO are simple and it is easy to calculate observer matrices. This class of observers can be, however, designed by Thoerem 3.1 as a special case. Letting $\hat{G}_i = \hat{G}$, $\forall\ i$ and $K_{ij} = K_i$, $\forall\ i,j$ in Theorem 3.1, we obtain the following corollary:

Corollary 3.1: A UIOED for the fuzzy bilinear system (1) and (2) is designed by (8) and (9) if there exist matrices $P_i > 0$, $\hat{G}$, $\hat{K}_i$, i $= 1,\cdots,r$ and S such that

$$\Phi_{ii} < 0,\ i = 1,\cdots,r \tag{33}$$

$$\frac{2}{r-1}\Phi_{ii} + \Phi_{ij} + \Phi_{ji} < 0,\ i,j,l = 1,\cdots,r,\ i \neq j \tag{34}$$

$$\Phi_{ij} = \begin{bmatrix} -P_i & * \\ (\hat{G}+S\bar{C})\bar{A}_i - \hat{K}_i\bar{C} & -\hat{G} - \hat{G}^T + P_j \end{bmatrix} < 0 \qquad i,j = 1,\cdots,r \tag{35}$$

$$(\hat{G}+S\bar{C})\bar{F}_i = 0,\ i = 1,\cdots,r \tag{36}$$

Observer matrices are calculated as

$$\hat{A}_i = (\hat{G}+S\bar{C})\bar{A}_i - \hat{K}_i\bar{C},\ i = 1\cdots,r \tag{37}$$

$$\hat{H} = \hat{G}^{-1}S,\ i = 1\cdots,r \tag{38}$$

$$\hat{L}_i = \hat{K}_i - \hat{A}_i\hat{H},\ i = 1\cdots,r \tag{39}$$

$$\hat{B}_i = (\hat{G}+S\bar{C})\bar{B}_i,\ i = 1\cdots,r \tag{40}$$

$$\hat{M}_i = (\hat{G}+S\bar{C})(\bar{M}_i\bar{C}+\bar{N}_i)\bar{C}^{+},\ i = 1\cdots,r \tag{41}$$

Proof: If $P_{ij} = P_i$, $\hat{G}_i = G$, $K_{ij} = K_i$, $i,j = 1,\cdots,r$, then the condition (12) becomes

$$\begin{bmatrix} -P_i & * \\ \hat{G}T\bar{A}_i - \hat{K}_i\bar{C} & -\hat{G} - \hat{G}^T + P_j \end{bmatrix} < 0,\ \forall\ i,j$$

Since T satisfies (19), the above LMIs can be written as (35) where

$$S = \hat{G}\hat{H} \tag{42}$$

(36) follows from the condition (13). (37)-(41) obviously follow from (14)-(19) and (42).

C. Extension to Higher-Order Disturbances

In (4), we consider the case where the disturbance d_{2k} is constant. However, it may not always be true in general cases. Here we consider the n-th order continuous-time disturbances such as

$$d_2(t) = \alpha_n t^n + \alpha_{n-1}t^{n-1} + \ldots + \alpha_0$$

By introduction of

$$\begin{aligned} \eta_0(t) &= d_2(t) \\ \eta_1(t) &= \frac{dd_2(t)}{dt} \\ &\vdots \\ \eta_{n-1}(t) &= \frac{d^{n-1}d_2(t)}{dt^{n-1}} \end{aligned}$$

$d_2(t)$ can be written as

$$\dot{\eta}(t) = \begin{bmatrix} \dot{\eta}_0(t) \\ \dot{\eta}_1(t) \\ \vdots \\ \dot{\eta}_{n-2}(t) \\ \dot{\eta}_{n-1}(t) \end{bmatrix} = \Gamma\eta(t)$$

$$d_2(t) = \Delta\eta(t)$$

where $\eta(t) = [\eta_0(t) \quad \eta_1(t) \quad \cdots \quad \eta_{n-2}(t) \quad \eta_{n-1}(t)]^T$ and

$$\Gamma = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix},\ \Delta = [1 \quad 0 \quad 0 \quad \ldots \quad 0]$$

This can be discretized with the sampling period T_d as

$$\eta_{k+1} = \Gamma_d\eta_k, d_{2k} = \Delta\eta_k \tag{43}$$

where $\Gamma_d = e^{\Gamma T_d}$. Sampling time can be taken as the same as the one such that system is discretized. Combining (1), (2), (5) and (43), we have

$$\bar{x}_{k+1} = \bar{A}_\lambda\bar{x}_k + \bar{B}_\lambda u_k + \bar{M}_\lambda y_k u_k + N_\lambda\bar{x}_k u_k + \bar{F}_\lambda d_{1k} \tag{44}$$

$$y_k = \bar{C}\bar{x}_k \tag{45}$$

where $\bar{x}_k = [x_k^T \quad \eta_k^T \quad d_{3k}^T]^T$ and

$$\bar{A}_\lambda = \begin{bmatrix} A_\lambda & -B_\lambda\Delta & 0 \\ 0 & \Gamma_d & 0 \\ 0 & 0 & I \end{bmatrix},\ \bar{B}_\lambda = \begin{bmatrix} B_\lambda \\ 0 \\ 0 \end{bmatrix},\ \bar{M}_\lambda = \begin{bmatrix} M_\lambda \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{N}_\lambda = \begin{bmatrix} N_\lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \bar{F}_\lambda = \begin{bmatrix} F_\lambda \\ 0 \\ 0 \end{bmatrix},\ \bar{C} = [C \quad 0 \quad -I]$$

Since the system (44) and (45) has the same form as (6) and (7), we can apply Theorem 3.1 with the observer form (8) and

$$\hat{\eta}_k = z_k - \hat{H} y_k \tag{46}$$

$$\hat{d}_{2k} = \Delta \hat{\eta}_k \tag{47}$$

to get the following theorem.

Theorem 3.2: A UIOED for the system (44) and (45) is given by (8), (9), (46) and (47) if there exist matrices $P_i > 0$, $\hat{G}_i$, $\hat{K}_{ij}$, $i, j = 1, \cdots, r$ and T such that

$$\Phi_{ii}^l < 0, \ i, l = 1, \cdots, r \tag{48}$$

$$\frac{2}{r-1}\Phi_{ii}^l + \Phi_{ij}^l + \Phi_{ji}^l < 0, \ i, j, l = 1, \cdots, r, \ i \neq j \tag{49}$$

$$\Phi_{ij}^l = \begin{bmatrix} -P_l & * \\ \hat{G}_j T \bar{A}_i - \hat{K}_{ij}\bar{C} & -\hat{G}_j - \hat{G}_j^T + P_i \end{bmatrix} \tag{50}$$

$$T\bar{F}_i = 0, \ i = 1, \cdots, r \tag{51}$$

Observer gain matrices are obtained by

$$\hat{A}_{ij} = \hat{G}_j T \bar{A}_i - \hat{K}_{ij} C, \ i, j = 1 \cdots, r \tag{52}$$

$$\hat{H} = (T - I)\bar{C}^+ \tag{53}$$

$$\hat{L}_{ij} = \hat{K}_{ij} - \hat{A}_{ij}\hat{H}, \ i, j = 1 \cdots, r \tag{54}$$

$$\hat{B}_{ij} = \hat{G}_j T \bar{B}_i, \ i, j = 1 \cdots, r \tag{55}$$

$$\hat{M}_{ij} = \hat{G}_j T (\bar{M}_i \bar{C} + \bar{N}_i)\bar{C}^+, \ i, j = 1 \cdots, r \tag{56}$$

where C^+ is a pseudo-inverse matrix of C.

Remark 3.3: Similar to d_{2k}, d_{3k} can be extended to n-th order disturbance and can be written as in (43). Consequently, a UIOED is designed by a straightforward extension of Theorem 3.2.

IV. NUMERICAL EXAMPLE

Let us consider the discrete-time fuzzy bilinear system described by (1), (2) and (4) with

$$A_1 = \begin{bmatrix} 2.1 & -0.1 \\ 0.8 & -0.5 \end{bmatrix}, \ A_2 = \begin{bmatrix} 2.0 & 0.4 \\ 1.3 & 0.7 \end{bmatrix}, \ B_1 = \begin{bmatrix} 1.2 \\ 1.0 \end{bmatrix}$$
$$B_2 = \begin{bmatrix} 0.9 \\ 1.1 \end{bmatrix}, \ M_1 = M_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ N_1 = \begin{bmatrix} 1.2 & 0 \\ 0 & -0.3 \end{bmatrix}$$
$$N_2 = \begin{bmatrix} 1.0 & 0 \\ 0 & -0.2 \end{bmatrix}, \ F_1 = \begin{bmatrix} 0.6 \\ 1 \end{bmatrix}, \ F_2 = \begin{bmatrix} 0.3 \\ 0.5 \end{bmatrix}$$
$$C = [1.0 \quad 0]$$

and the membership functions are given by

$$\lambda_1(x_1) = \frac{1 + \cos(x_1)}{2}, \ \lambda_2(x_1) = 1 - \lambda_1(x_1)$$

For this system, the conditions of Corollary 3.1 and other PDO design conditions are not met, and hence it is impossible to make a PDO design. On the other hand, Theorem 3.1, which employs non-PDO, can provide observer matrices of the observer (8) and (9). For the limited space, only some observer matrices for this example are provided;

$$P_1 = \begin{bmatrix} 3.4715 & 0.0676 & -0.4722 & 0.5699 \\ 0.0676 & 0.6253 & -0.0487 & 0.0676 \\ -0.4722 & -0.0487 & 3.3007 & -0.4722 \\ 0.5699 & 0.0676 & -0.4722 & 3.4715 \end{bmatrix}$$
$$\hat{A}_{11} = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ -1.1770 & -0.2906 & 0.8719 & -1.1770 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$
$$\hat{A}_{22} = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ -0.8864 & -0.0291 & 0.8719 & -0.8864 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$
$$\hat{B}_{11} = \begin{bmatrix} 0.0 \\ -0.8719 \\ 0.0 \\ 0.0 \end{bmatrix}, \ \hat{B}_{22} = \begin{bmatrix} 0.0 \\ -0.3488 \\ 0.0 \\ 0.0 \end{bmatrix}$$
$$\hat{M}_{11} = \begin{bmatrix} 0.0 \\ -0.8719 \\ 0.0 \\ 0.0 \end{bmatrix}, \ \hat{M}_{22} = \begin{bmatrix} 0.0 \\ -0.7266 \\ 0.0 \\ 0.0 \end{bmatrix}$$
$$\hat{L}_{11} = \begin{bmatrix} 0.0 \\ -1.4192 \\ 0.0 \\ 0.0 \end{bmatrix}, \ \hat{L}_{22} = \begin{bmatrix} 0.0 \\ -0.8622 \\ 0.0 \\ 0.0 \end{bmatrix}$$
$$\hat{H} = \begin{bmatrix} -0.5 \\ -0.8333 \\ 0.0 \\ 0.5 \end{bmatrix}$$

With the initial conditions $e(0) = [-1.0 \ 0.5 \ 1.0 \ 2.0]^T$ and the inputs $d_{1k} = 0.3\cos(k)$, $d_{2k} = 1.0$, $d_{3k} = 2.0$, the error e_k between the actual states and their estimates is simulated and is shown in Figure 1. The lines indicate the errors of x_1, x_2, d_2 and d_3, respectively. Those lines clearly show that our UIOED estimates the true values of the states x_1, x_2 and disturbances d_2, d_3 because the errors of x_1, x_2, d_2 and d_3 diminish as k goes to infinity.

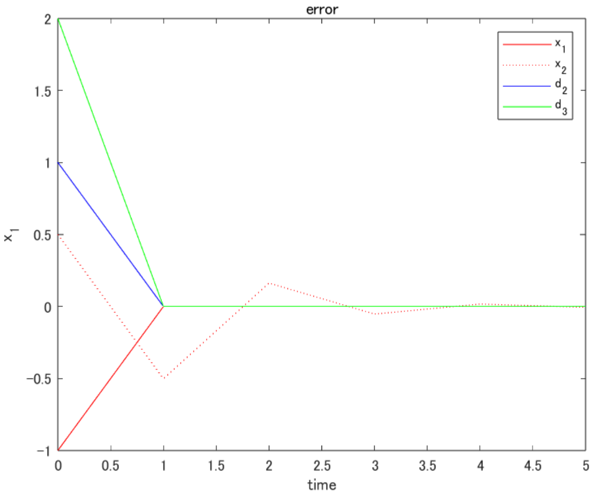

Fig. 1. The error trajectories.

V. CONCLUSIONS

A UIOED design problem of Takagi-Sugeno fuzzy bilinear systems has been solved. A UIOED makes it possible to estimate the state variables with filtering out unknown inputs. A new fuzzy Lyapunov function was employed to show improved observer design conditions. Our observer design conditions were shown to be less conservative. Using a numerical example We also showed our UIOED estimates the state variables and input/output disturbances and removes unknown inputs.

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