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Improved Observer Design with Time-Delay Membership Functions for Takagi-Sugeno Fuzzy Systems

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Abstract— **The state estimation is important issues in system theory and control systems. Especially, estimation of the state variables of general nonlinear systems is essential but it is still difficult at the same time. This paper proposes design methods for an observer of a nonlinear system described by Takagi-Sugeno(T-S) fuzzy system. Because of the capability of representation of T-S fuzzy system, our observers are designed for a quite large class of nonlinear systems, which cover most systems in various engineering fields. Our observer design employs multiple Lyapunov matrix methods, which result from Lyapunov function candidate with the multiple integrals of the membership functions. Our resulting observers are a non-parallel distributed observer (PDO). This method drastically reduces the conservativeness of observer design conditions. In order to provide the usefulness of our proposed design approach, an illustrative example is provided. Finally, we end with concluding remarks.**

I. INTRODUCTION

In real societies, almost all practical systems are described by nonlinear models, and the state variables of the system are not always measured. Due to this fact, observers with the measurable inputs and outputs are designed to estimate the state variables of nonlinear systems. For linear systems, observer theory has been established. On the other hand, a design of observers for nonlinear systems is a quite difficult problem. About fourty years ago, T-S fuzzy system was discovered to represent a wide class of the nonlinear systems([9]). T-S fuzzy system is described by a polytopic type of linear systems, and is well-known global representation of nonlinear systems. Due to this, T-S fuzzy system has been often employed for analysis and synthesis of nonlinear systems([1], [2], [5], [6], [8], [10], [11], [12], [13], [14], [15], [16], [17], and [18]). The stability analysis and control design for T-S fuzzy systems were first considered in [12] where the concept of the parallel distributed compensator(PDC) with stability and control design conditions was introduced to proposed the state feedback controller. The PDC also has a polytopic structure and has been frequently utilized for T-S fuzzy systems for more than three decades since then. First stability conditions and controller design conditions via PDC were found in [12] and [13]. Then, the papers [2], [14], and [15] have tried to reduce the conservativeness of stability and stabilizability conditions via PDC by different kinds of relaxation methods for improvemnts. The paper [11] found out that descriptor system representation leads to further less conservative conditions on stability and control design. In addition, a multiple Lyapunov matrix method([6] and [10]) and a line-integral method([5] and [8]) have appeared for further reduction of the conservatism in control design methods. In recent years, a quite totally different approach from the conventional PDC, that is, a non-PDC, which reduces a large amount of the conservativeness in stability and control design conditions, was considered([1], [5], and [6]). Not only the state feedback controller design, but also observer design began. The classical observer for T-S fuzzy systems is the parallel distributed observer(PDO), which has also a polytopic structure introduced in [4], [13], [17], and [18]. The PDO is designed by the same idea of the PDC. The PDO is a convex combination of observers which are constructed for each local linear system in a fuzzy system, and uses the same weighting functions as those of the system. The paper [3] employed a different class of observer from PDO and reduced conservativeness, but there is still room to be improved.

Similar to the state feedback control design, a multiple Lyapunov matrix method reduces the conservativeness in observer design. However, the upper bounds of the derivative of each membership function must to be known to satisfy the design conditions. This restriction causes the infeasibility of the design method because some membership function are not differentiable and hence its derivative does not exist. To overcome this flaw, a different approach with different multiple Lyapunov matrices was considered in [6]. In there, the Lyapunov function candidate has an integral of the membership function. An approach with this class of the Lyapunov functions indeed overcome the membership function differentiability issue. Besides, this approach reduces more conservativeness in control and observer design conditions than the conventional single Lyapunov matrix approach. Further improvement was made in [19] and [21]. However, this approach has not yet been utilized to the observer issues for T-S fuzzy systems.

This paper extensively discuss the observer design problem for T-S fuzzy systems, and propose a new design approach. Apart from the traditional method such as PDO, a type class of non-PDOs is used. We use the weighting functions of observer gains which are integrals of the membership functions. Correspondingly, a non-quadratic Lyapunov function with multiple Lyapunov matrix and integrals of the membership functions is selected to obtain new observer design conditions such that the estimation error is asymptotically stable. In addition, the integral intervals are divided into small intervals, and more slack matrices are introduced. Furthermore, some relaxation lemmas are

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applied to reduce conservativeness from observer design conditions. These approaches lead to more flexible in observer design conditions and consequently, our proposed design methods make observe design for a wider class of T-S fuzzy system possible than the previous works. For calculation friendliness, the proposed observer design conditions are basically given in linear matrix inequalities(LMIs), and is solvable easily by a mathematical software. An observer design example is shown to give the effectiveness of our observer design methods proposed here.

In this paper, the following notation is employed:

$$P_\omega = \sum_{i=1}^{r} \omega_i(\zeta(t))P_i,\ P_{\omega^h} = \sum_{i=1}^{r} \omega_i(\zeta(t-h))P_i$$

$$P_{\omega\omega} = \sum_{i=1}^{r}\sum_{j=1}^{r} \omega_i(\zeta(t))\omega_j(\zeta(t))P_{ij}$$

II. T-S FUZZY SYSTEMS

This first section explains a class of T-S fuzzy systems under consideration of the paper.

The T-S fuzzy model is expressed by the following fuzzy if-then rules:

$$\begin{array}{ll} \textbf{IF} & \zeta_1 \text{ is } S_{i1} \text{ and } \cdots \text{ and } \zeta_p \text{ is } S_{ip} \\ \textbf{THEN} & \dot{x}(t) = A_i x(t) + B_i u(t) \\ & y(t) = C_i x(t),\ \forall\ i = 1,\cdots,r \end{array}$$

where $\zeta_i,\ i = 1,\cdots,p$ and S_{ij} are the premise variables and fuzzy sets, respectively. $x(t) \in \Re^n$, $u(t) \in \Re^m$, and $y(t) \in \Re^q$ are the state variable, the input, respectively. A_i, B_i and C_i denote constant matrices of appropriate dimensions. r is the number of if-then rules. For simplicity, we denote $\zeta = [\,\zeta_1 \quad \cdots \quad \zeta_p\,]^T$, and the premise variables $\zeta_i,\ i = 1,\ldots,p$ are assumed to be given.

The overall state and the output equations in the T-S fuzzy system are described by

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{r} \omega_i(\zeta)\{A_i x(t) + B_i u(t)\} \\ &\overset{def}{=} A_\omega x(t) + B_\omega u(t) && (1) \\ y(t) &= \sum_{i=1}^{r} \omega_i(\zeta) C_i x(t) \\ &\overset{def}{=} C_\omega x(t) && (2) \end{aligned}$$

respectively, where

$$\omega_i(\zeta) = \frac{\alpha_i(\zeta)}{\sum_{i=1}^{r} \alpha_i(\zeta)},\quad \alpha_i(\zeta) = \prod_{j=1}^{p} S_{ij}(\zeta_j) \qquad (3)$$

and $S_{ij}(\cdot)$ is the grade of the membership function of S_{ij}. It is assumed that

$$\alpha_i(\zeta(t)) \geq 0,\ i = 1,\cdots,r,\ \sum_{i=1}^{r} \alpha_i(\zeta(t)) > 0$$

for all $\zeta(t)$. Consequently, $\omega_i(\zeta(t))$ satisfies

$$\omega_i(\zeta(t)) \geq 0,\ i = 1,\cdots,r,\ \sum_{i=1}^{r} \omega_i(\zeta(t)) = 1$$

for all $\zeta(t)$.

Now, the problem in this paper is to design an observer that estimates the state of the system (1) and (2).

Before we complete this section, the following lemmas are introduced to derive our results in the later sections.

Lemma 2.1: ([15])

$$\sum_{i=1}^{r}\sum_{j=1}^{r} \omega_i(\zeta_k)\omega_j(\zeta_k)\Phi_{ij} < 0$$

holds if the following is satisfied.

$$\begin{aligned} \Phi_{ii} < 0, &\qquad i = 1,\cdots,r \\ \frac{2}{r-1}\Phi_{ii} + \Phi_{ij} + \Phi_{ji} < 0, &\quad i \neq j,\ i,j = 1,\cdots,r \end{aligned}$$

Lemma 2.2: ([6]) Let A, R, L, P and Q be matrices of appropriate dimensions. The following inequalities are equivalent.

1) A and $Q > 0$ are given. There exists $P > 0$ such that

$$A^T P + PA + Q < 0$$

2) A and $Q > 0$ are given. There exist $P > 0$, L and R such that

$$\begin{bmatrix} A^T L^T + LA + Q & P - L - A^T R \\ P - L^T + R^T A & -R - R^T \end{bmatrix} < 0$$

III. FUZZY OBSERVER DESIGN

In this section, we will propose design methods for the observer problem formulated in Section II. Based on new asymptotic stability conditions of the estimation error between the actual state and its estimate via multiple Lyapunov matrix approach, we shall obtain less conservative design conditions of an observer for the fuzzy system (1) and (2) than the previous works.

A. Non-PDO Design

For a quite long time, the parallel distributed observer (PDO) has been used for fuzzy systems. Observer design conditions via PDO, however, are generally restrictive and are infeasible to design an observer for a large class of T-S fuzzy systems. To avoid such a disadvantage, we will propose the following non-parallel distributed observer (non-PDO):

$$\begin{aligned} \dot{\hat{x}}(t) &= A_\omega \hat{x}(t) + B_\omega u(t) \\ &\quad + P_\mu^{-1} K_{\omega\mu\omega^h\omega^{2h}}(y(t) - \hat{C}_\omega \hat{x}(t)) && (4) \end{aligned}$$

where $\hat{x}(t) \in \Re^n$ is the estimate of $x(t)$,

$$P_\mu = P_{1\mu} + P_{2\mu},\ K_{\mu\omega^h\omega^{2h}} = K_{1\mu\omega^h\omega^{2h}} + K_{2\mu\omega^h\omega^{2h}}$$

$$P_{1\mu} = \sum_{i=1}^{r} \mu_{1i}(t)P_{1i},\ P_{2\mu} = \sum_{i=1}^{r} \mu_{2i}(t)P_{2i}$$

$$K_{1\mu\omega^h\omega^{2h}} = \sum_{j=1}^{r}\sum_{k=1}^{r}\sum_{l=1}^{r} \mu_j(t)\omega_k(t-h)\omega_l(t-2h)K_{1ijk}$$

$$K_{2\mu\omega^h\omega^{2h}} = \sum_{j=1}^{r}\sum_{k=1}^{r}\sum_{l=1}^{r} \mu_j(t)\omega_k(t-h)\omega_l(t-2h)K_{2ijk}$$

The variables $\mu_{1i}(t)$ and $\mu_{2i}(t)$ are given by

$$\mu_{1i}(t) = \frac{1}{h}\int_{t-h}^{t} \omega_i(\zeta(s))ds$$

$$\mu_{2i}(t) = \frac{1}{h}\int_{t-2h}^{t-h} \omega_i(\zeta(s))ds$$

with some scalar $h > 0$. $P_{1i},\ P_{2i},\ K_{1ijk}$ and K_{2ijk} are observer gain matrices to be designed.

Now, the following is our first result in this research.

Theorem 3.1: An observer for the system (1) and (2) is given by (4) if for some $h > 0$ there exist matrices $P_{1j} > 0, P_{2j} > 0,\ K_{1jkl},\ K_{2jkl},\ j,k,l = 1,\cdots,r$ such that

$$\Phi_{1ii}^{kl} < 0,\ i,k,l = 1,\cdots,r \qquad (5)$$

$$\frac{2}{r-1}\Phi_{1ii}^{kl} + \Phi_{1ij}^{kl} + \Phi_{1ji}^{kl} < 0,\ i,j,k,l = 1,\cdots,r,\ i \neq j \qquad (6)$$

$$\Phi_{2ii}^{kl} < 0,\ i,k,l = 1,\cdots,r \qquad (7)$$

$$\frac{2}{r-1}\Phi_{2ii}^{kl} + \Phi_{2ij}^{kl} + \Phi_{2ji}^{kl} < 0,\ i,j,k,l = 1,\cdots,r,\ i \neq j \qquad (8)$$

$$\begin{aligned}
\Phi_{1ij}^{kl} &= P_{1j}A_i + A_i^T P_{1j} - K_{1jkl}C_i - C_i^T K_{1jkl}^T \\
&\quad + \frac{1}{2h}(P_{1i} - P_{1k} + P_{2k} - P_{2l}) \\
\Phi_{2ij}^{kl} &= P_{2j}A_i + A_i^T P_{2j} - K_{2jkl}C_i - C_i^T K_{2jkl}^T \\
&\quad + \frac{1}{2h}(P_{1i} - P_{1k} + P_{2k} - P_{2l})
\end{aligned}$$

Observer gain matrices $P_{1j},\ P_{2j},\ K_{1jkl}$ and K_{2jkl} can be obtained in the above LMIs.

Proof: To design of an observer of the form (4), we investigate the error $\gamma(t)$ between $x(t)$ and $\hat{x}(t)$;

$$\gamma(t) = x(t) - \hat{x}(t)$$

The error $\gamma(t)$ satisfies

$$\dot{\gamma}(t) = (A_\omega - P_\mu^{-1}K_{\mu\omega^h\omega^{2h}}C_\omega)\gamma(t) \qquad (9)$$

If the estimation error (9) is asymptotically stable, (4) becomes an observer for the system (1) and (2).

Now, we investigate a condition for (9) to be asymptotically stable. For that purpose, we take the following Lypunov function candidate:

$$\begin{aligned}
W(\gamma(t)) &= \gamma^T(t)P_\mu\gamma(t) \\
&= \gamma^T(t)(\sum_{i=1}^{r}\mu_{1i}P_{1i} + \sum_{i=1}^{r}\mu_{2i}P_{2i})^{-1}\gamma(t) \qquad (10)
\end{aligned}$$

where $P_{1i} > 0,\ P_{2i} > 0\ i = 1,\cdots,r$ to be determined. The variables $\mu_{1i}(t)$ and $\mu_{2i}(t)$ have the following properties:

$$\dot{\mu}_{1i}(t) = \frac{1}{h}(\omega_i(\zeta(t)) - \omega_i(\zeta(t-h)))$$

$$\dot{\mu}_{2i}(t) = \frac{1}{h}(\omega_i(\zeta(t-h)) - \omega_i(\zeta(t-2h)))$$

respectively. In addition, we have

$$\begin{aligned}
\sum_{i=0}^{r}\mu_{1i} &= \frac{1}{h}\int_{t-h}^{t}\sum_{i=0}^{r}\omega_i(\zeta(s))ds \\
&= \frac{1}{h}\int_{t-h}^{t} 1ds \\
&= 1 \\
\sum_{i=0}^{r}\mu_{2i} &= \frac{1}{h}\int_{t-2h}^{t-h}\sum_{i=0}^{r}\omega_i(\zeta(s))ds \\
&= \frac{1}{h}\int_{t-2h}^{t-h} 1ds \\
&= 1
\end{aligned}$$

To derive a new asymptotic stability condition for (9), we use Lyapunov stability theorem. Calculate the time-derivative of $W(\gamma(t))$ along the solution of (9):

$$\begin{aligned}
\dot{W}(\gamma(t)) &= 2\gamma^T(t)(\sum_{i=1}^{r}\mu_{1i}P_{1i} + \sum_{i=1}^{r}\mu_{2i}P_{2i})\dot{\gamma}(t) \\
&\quad + \gamma^T(t)(\sum_{i=1}^{r}\dot{\mu}_{1i}P_{1i} + \sum_{i=1}^{r}\dot{\mu}_{2i}P_{2i})\gamma(t) \\
&= 2\gamma^T(t)(P_{1\mu} + P_{2\mu})(A_\omega - P_\mu^{-1}K_{\omega\mu\omega^h\omega^{2h}}C_\omega)\gamma(t) \\
&\quad + \frac{1}{h}\gamma^T(t)(P_{1\omega} - P_{1\omega^h} + P_{2\omega} - P_{2\omega^h})\gamma(t) \\
&= \gamma^T(t)\{P_\mu A_\omega - K_{\omega\mu\omega^h\omega^{2h}}C_\omega \\
&\quad + A_\omega^T P_\mu - C_\omega^T K_{\omega\mu\omega^h\omega^{2h}}^T \\
&\quad + \frac{1}{h}(P_{1\omega} - P_{1\omega^h} + P_{2\omega} - P_{2\omega^h})\}\gamma(t) \\
&= \gamma^T(t)\{P_{1\mu}A_\omega + A_\omega^T P_{1\mu} \\
&\quad - K_{1\mu\omega^h\omega^{2h}}C_\omega - C_\omega^T K_{1\mu\omega^h\omega^{2h}}^T \\
&\quad + \frac{1}{2h}(P_{1\omega} - P_{1\omega^h} + P_{2\omega^h} - P_{2\omega^{2h}})\}\gamma(t) \\
&\quad + \gamma^T(t)\{P_{2\mu}A_\omega + A_\omega^T P_{2\mu} \\
&\quad - K_{2\mu\omega^h\omega^{2h}}C_\omega - C_\omega^T K_{2\mu\omega^h\omega^{2h}}^T \\
&\quad + \frac{1}{2h}(P_{1\omega} - P_{1\omega^h} + P_{2\omega^h} - P_{2\omega^{2h}})\}\gamma(t) \qquad (11)
\end{aligned}$$

where

$$P_{1\omega^h} = \sum_{i=1}^{r}\omega_i(\zeta(t-h))P_{1i},\ P_{2\omega^{2h}} = \sum_{i=1}^{r}\omega_i(\zeta(t-2h))P_{2i}$$

Obviously, if

$$\begin{aligned}
&P_{1\mu}A_\omega + A_\omega^T P_{1\mu} - K_{1\mu\omega^h\omega^{2h}}C_\omega - C_\omega^T K_{1\mu\omega^h\omega^{2h}}^T \\
&\quad + \frac{1}{2h}(P_{1\omega} - P_{1\omega^h} + P_{2\omega^h} - P_{2\omega^{2h}}) < 0 \qquad (12) \\
&P_{2\mu}A_\omega + A_\omega^T P_{2\mu} - K_{2\mu\omega^h\omega^{2h}}C_\omega - C_\omega^T K_{2\mu\omega^h\omega^{2h}}^T \\
&\quad + \frac{1}{2h}(P_{1\omega} - P_{1\omega^h} + P_{2\omega^h} - P_{2\omega^{2h}}) < 0 \qquad (13)
\end{aligned}$$

are satisfied, we have $\dot{W}(\gamma(t)) < 0$, which implies that the estimation error (9) is asymptotically stable. Therefore, in this case (4) becomes an observer for the T-S fuzzy system

(1) and (2). The conditions (12) and (13) can be rewritten as

$$\begin{aligned}\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r}\sum_{l=1}^{r}&\omega_i(\zeta(t))\mu_j(\zeta(t))\omega_k(\zeta(t-h))\omega_l(\zeta(t-2h))\\ &\times\{P_{1j}A_i+A_i^TP_{1j}-K_{1jkl}C_i-C_i^TK_{1jkl}^T\\ &+\frac{1}{2h}(P_{1i}-P_{1k}+P_{2k}-P_{2l})\}<0\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r}\sum_{l=1}^{r}&\omega_i(\zeta(t))\mu_j(\zeta(t))\omega_k(\zeta(t-h))\omega_l(\zeta(t-2h))\\ &\times\{P_{2j}A_i+A_i^TP_{2j}-K_{2jkl}C_i-C_i^TK_{2jkl}^T\\ &+\frac{1}{2h}(P_{1i}-P_{1k}+P_{2k}-P_{2l})\}<0\end{aligned}$$

Finally, using Lemma 2.1, we get (5)-(8).

Letting $K_{1jkl}=K_{1j}$ and $K_{2jkl}=K_{2j}$ in Theorem 3.1, we have the following corollary, which requires less number of matrix variables so that computational burden is reduced from Theorem 3.1 but the corollary still preserve effectiveness to design an observer because it basically follows from the Lyapunov function candidate (10).

Corollary 3.1: An observer for the system (1) and (2) is given by (18) if for some $h>0$ there exist matrices $P_{1j}>0, P_{2j}>0,\ K_{1j},\ K_{2j}\ j=1,\cdots,r$ such that

$$\Phi_{1ii}^{kl}<0,\ i,k,l=1,\cdots,r \qquad (14)$$

$$\frac{2}{r-1}\Phi_{1ii}^{kl}+\Phi_{1ij}^{kl}+\Phi_{1ji}^{kl}<0,\ i,j,k,l=1,\cdots,r,\ i\neq j \qquad (15)$$

$$\Phi_{2ii}^{kl}<0,\ i,k,l=1,\cdots,r \qquad (16)$$

$$\frac{2}{r-1}\Phi_{2ii}^{kl}+\Phi_{2ij}^{kl}+\Phi_{2ji}^{kl}<0,\ i,j,k,l=1,\cdots,r,\ i\neq j \qquad (17)$$

$$\begin{aligned}\Phi_{1ij}^{kl}&=P_{1j}A_i+A_i^TP_{1j}-K_{1j}C_i-C_i^TK_{1j}^T\\ &\quad+\frac{1}{2h}(P_{1i}-P_{1k}+P_{2k}-P_{2l})\\ \Phi_{2ij}^{kl}&=P_{2j}A_i+A_i^TP_{2j}-K_{2j}C_i-C_i^TK_{2j}^T\\ &\quad+\frac{1}{2h}(P_{1i}-P_{1k}+P_{2k}-P_{2l})\end{aligned}$$

Observer gain matrices P_{1j}, P_{2j}, K_{1j} and K_{2j} can be obtained in the above LMIs, and an observer is given by

$$\begin{aligned}\dot{\hat{x}}(t)\ &=\ A_\omega\hat{x}(t)+B_\omega u(t)\\ &\quad+P_\mu^{-1}K_\omega(y(t)-\hat{C}_\omega\hat{x}(t))\end{aligned} \qquad (18)$$

where

$$K_\omega=K_{1\omega}+K_{2\omega}$$

$$K_{1\omega}=\sum_{i=1}^{r}\omega_i(\zeta(t))K_{1i},\ K_{2\omega}=\sum_{i=1}^{r}\omega_i(\zeta(t))K_{2i}$$

The following is our second main result:

Theorem 3.2: An observer for the system (1) and (2) is given by (4) if for some $h>0$ there exist matrices $P_{1j}>0, P_{2j}>0,\ K_{1jkl},\ K_{2jkl},\ L_{1jkl},\ L_{2jkl},\ R_{1jkl},\ R_{2jkl},\ i,j,k,l=1,\cdots,r$ such that

$$\Phi_{1ii}^{kl}<0,\ i,k,l=1,\cdots,r \qquad (19)$$

$$\frac{2}{r-1}\Phi_{1ii}^{kl}+\Phi_{1ij}^{kl}+\Phi_{1ji}^{kl}<0,\ i,j,k,l=1,\cdots,r,\ i\neq j \qquad (20)$$

$$\Phi_{2ii}^{kl}<0,\ i,k,l=1,\cdots,r \qquad (21)$$

$$\frac{2}{r-1}\Phi_{2ii}^{kl}+\Phi_{2ij}^{kl}+\Phi_{2ji}^{kl}<0,\ i,j,k,l=1,\cdots,r,\ i\neq j \qquad (22)$$

$$\begin{aligned}\Phi_{1ij}^{kl}&=\begin{bmatrix}\Phi_{11ij}^{kl} & P_{1j}-L_{1jkl}+A_i^TR_{1jkl}\\ * & -R_{1jkl}-R_{1jkl}^T\end{bmatrix}\\ \Phi_{2ij}^{kl}&=\begin{bmatrix}\Phi_{22ij}^{kl} & P_{2j}-L_{2jkl}+A_i^TR_{2jkl}\\ * & -R_{2jkl}-R_{2jkl}^T\end{bmatrix}\\ \Phi_{11ij}^{kl}&=L_{1jkl}A_i+A_i^TL_{1jkl}^T-K_{1jkl}C_i-C_i^TK_{1jkl}^T\\ &\quad+\frac{1}{2h}(P_{1i}-P_{1k}+P_{2k}-P_{2l})\\ \Phi_{22ij}^{kl}&=L_{2jkl}A_i+A_i^TL_{2jkl}^T-K_{2jkl}C_i-C_i^TK_{2jkl}^T\\ &\quad+\frac{1}{2h}(P_{1i}-P_{1k}+P_{2k}-P_{2l})\end{aligned}$$

Observer gain matrices P_{1j}, P_{2j}, K_{1jkl} and K_{2jkl} can be obtained in the above LMIs.

Proof: Direct application of Lemma 2.2 to Theorem 3.1 leads to the desired result.

Similar to Corollary 3.1, letting $K_{1jkl}=K_{1j}$ and $K_{1jkl}=K_{1j}$ in Theorem 3.2, we get the following result.

Corollary 3.2: An observer for the system (1) and (2) is given by (4) if for some $h>0$ there exist matrices $P_{1j}>0$, $P_{2j}>0$, K_{1j}, K_{2j} L_{1jkl}, L_{2jkl}, R_{1jkl}, R_{2jkl}, $j,k,l=1,\cdots,r$ such that

$$\Phi_{1ii}^{kl}<0,\ i,k,l=1,\cdots,r \qquad (23)$$

$$\frac{2}{r-1}\Phi_{1ii}^{kl}+\Phi_{1ij}^{kl}+\Phi_{1ji}^{kl}<0,\ i,j,k,l=1,\cdots,r,\ i\neq j \qquad (24)$$

$$\Phi_{2ii}^{kl}<0,\ i,k,l=1,\cdots,r \qquad (25)$$

$$\frac{2}{r-1}\Phi_{2ii}^{kl}+\Phi_{2ij}^{kl}+\Phi_{2ji}^{kl}<0,\ i,j,k,l=1,\cdots,r,\ i\neq j \qquad (26)$$

$$\begin{aligned}\Phi_{1ij}^{kl}&=\begin{bmatrix}\Phi_{11ij}^{kl} & P_{1j}-L_{1jkl}+A_i^TR_{1jkl}\\ * & -R_{1jkl}-R_{1jkl}^T\end{bmatrix}\\ \Phi_{2ij}^{kl}&=\begin{bmatrix}\Phi_{22ij}^{kl} & P_{2j}-L_{2jkl}+A_i^TR_{2jkl}\\ * & -R_{2jkl}-R_{2jkl}^T\end{bmatrix}\\ \Phi_{11ij}^{kl}&=L_{1jkl}A_i+A_i^TL_{1jkl}^T-K_{1j}C_i-C_i^TK_{1j}^T\\ &\quad+\frac{1}{2h}(P_{1i}-P_{1k}+P_{2k}-P_{2l})\\ \Phi_{22ij}^{kl}&=L_{2jkl}A_i+A_i^TL_{2jkl}^T-K_{2j}C_i-C_i^TK_{2j}^T\\ &\quad+\frac{1}{2h}(P_{1i}-P_{1k}+P_{2k}-P_{2l})\end{aligned}$$

Observer gain matrices P_{1j}, P_{2j}, K_{1j} and K_{2j} can be obtained in the above LMIs, and an observer is given by (18).

Remark 3.1: Corollaries 3.1 and 3.2 reduce a large number of matrix variables K_{1jkl} and K_{2jkl} to be determied, but preserve the efficiency of the design methods.

If we take $P_{1i}=P_{2i}$, we recover the same result as in [20].

Corollary 3.3: An observer for the system (1) and (2) is given by (30) if for some $h>0$ there exist matrices $P_j>$

$0,\ K_{jk},\ j,k = 1,\cdots,r$ such that

$$\Phi_{ii}^{k} < 0,\ i,k = 1,\cdots,r \qquad (27)$$

$$\frac{2}{r-1}\Phi_{ii}^{k} + \Phi_{ij}^{k} + \Phi_{ji}^{k} < 0,\ i,j,k = 1,\cdots,r,\ i \neq j \qquad (28)$$

$$\Phi_{ij}^{k} = P_j A_i + A_i^T P_j - K_{jk} C_i - C_i^T K_{jk}^T + \frac{1}{2h}(P_i - P_k) \qquad (29)$$

Observer gain matrices P_j and K_{jk} can be obtained in the above LMIs, and an observer is given by

$$\begin{aligned}\dot{\hat{x}}(t) \;&=\; A_\omega \hat{x}(t) + B_\omega u(t)\\ &\quad + P_\mu^{-1} K_{\mu\omega^h}(y(t) - \hat{C}_\omega \hat{x}(t))\end{aligned} \qquad (30)$$

where

$$P_\mu = \sum_{i=1}^{r} \mu_{1i} P_i,$$

$$K_{\mu\omega^h} = \sum_{j=1}^{r}\sum_{k=1}^{r} \mu_j(t)\omega_k(t-2h)K_{jk}$$

Remark 3.2: Theorems 3.1 and 3.2 are generalized results of the previous works in [20] because the multiple Lyapunov matrix approach in (10) is adopted instead of the single Lyapunov matrix approach. This implies that (4) is an extended non-PDO of the previous observers. Actually, (4) recovers the PDO if $P_{ij} = P,\ i,j = 1,\cdots,r$, which is described in the next subsection.

B. PDO Design

A PDO has been widely used for T-S fuzzy systems. This type of observers can be reduced from Thoerem 3.1 as a special case. Letting $P_{1j} = P_{2j} = P,\ \forall\ j$ and $K_{1jkl} = K_{2jkl} = K_j,\ \forall\ j,k,l$ in Theorems 3.1 and 3.2, we readily get the following corollaries:

Corollary 3.4: An observer for the system (1) and (2) is given by (34) if for some $h > 0$ there exist matrices $P > 0,\ K_j,\ j = 1,\cdots,r$ such that

$$\Phi_{ii} < 0,\ i = 1,\cdots,r \qquad (31)$$

$$\frac{2}{r-1}\Phi_{ii} + \Phi_{ij} + \Phi_{ji} < 0,\ i,j = 1,\cdots,r,\ i \neq j \qquad (32)$$

$$\Phi_{ij} = PA_i + A_i^T P - K_j C_i - C_i^T K_j^T \qquad (33)$$

Observer gain matrices can be calculated in the above LMIs, and an observer is given by

$$\begin{aligned}\dot{\hat{x}}(t) \;&=\; A_\omega \hat{x}(t) + B_\omega u(t)\\ &\quad + P^{-1} K_\omega(y(t) - \hat{C}_\omega \hat{x}(t))\end{aligned} \qquad (34)$$

where

$$K_\omega = \sum_{i=1}^{r} \omega_i K_i$$

Corollary 3.5: An observer for the system (1) and (2) is given by (4) if for some $h > 0$ there exist matrices $P > 0,\ K_j,\ L_{ij},\ R_{ij},\ i,j = 1,\cdots,r$ and a scalar $h > 0$ such that

$$\Phi_{ii} < 0,\ i = 1,\cdots,r \qquad (35)$$

$$\frac{2}{r-1}\Phi_{ii} + \Phi_{ij} + \Phi_{ji} < 0,\ i,j = 1,\cdots,r,\ i \neq j \qquad (36)$$

$$\Phi_{ij} = \begin{bmatrix} \Phi_{11ij} & P - L_{1ij} + A_i^T R_{1ij} \\ * & -R_{ij} - R_{ij}^T \end{bmatrix}$$

Observer gain matrices K_j can be obtained in the above LMIs, and an observer is given by (34).

IV. NUMERICAL EXAMPLE

Now, consider the following fuzzy system described by (1) and (2) with matrices

$$\begin{aligned} A_1 &= \begin{bmatrix} 2.0 & 0.1 \\ 0 & -3.0 \end{bmatrix},\ A_2 = \begin{bmatrix} -2.0 & 3.0 \\ -0.1 & 3.0 \end{bmatrix} \\ B_1 &= \begin{bmatrix} 3.2 \\ 2.1 \end{bmatrix},\ B_2 = \begin{bmatrix} -1.1 \\ 1.2 \end{bmatrix} \\ C_1 &= \begin{bmatrix} -0.001 & 0.001 \end{bmatrix},\ C_2 = \begin{bmatrix} 0.01 & 1.0 \end{bmatrix} \end{aligned}$$

The membership functions are assumed to be

$$\omega_1(x_1) = \frac{1+\sin(x_1)}{2},\ \omega_2(x_1) = 1 - \omega_1(x_1)$$

For this example, none of the conditions of Corollary 3.2 in this paper, Theorem 3.1 in [20] and other PDO design has solutions, and hence these results cannot make an observer design for this system. However, Theorem 3.2 in this paper, which adopts non-PDO, offers observer gain matrices of the observer (4). Because the space is limited, only some observer gain matrices are listed here;

$$P_{11} = \begin{bmatrix} 11.0382 & 2.3529 \\ 2.3529 & 37.7039 \end{bmatrix},\ P_{12} = \begin{bmatrix} 26.1983 & 0.1129 \\ 0.1129 & 22.0155 \end{bmatrix}$$

$$P_{21} = \begin{bmatrix} 1.0380 & 2.3529 \\ 2.3529 & 37.7042 \end{bmatrix},\ P_{22} = \begin{bmatrix} 26.1981 & 0.1129 \\ 0.1129 & 22.0157 \end{bmatrix}$$

$$K_{1111} = \begin{bmatrix} 68.286 \\ 36.741 \end{bmatrix},\ K_{2222} = \begin{bmatrix} 56.4354 \\ 270.1376 \end{bmatrix}$$

$$L_{1111} = \begin{bmatrix} 28.8 & -2329.1 \\ -3510.5 & -101.2 \end{bmatrix}, R_{1111} = \begin{bmatrix} 9.6 & -1166.7 \\ 1170.8 & 8.1 \end{bmatrix}$$

$$L_{2222} = \begin{bmatrix} -4.6232 & 206.9118 \\ 343.1393 & -256.7631 \end{bmatrix}$$

$$R_{2222} = \begin{bmatrix} 10.4788 & -103.933 \\ 103.8651 & 11.0069 \end{bmatrix}$$

Given the initial conditions $e(0) = [2.0\ \ -1.0]^T$, the trajectory of the estimation error $e(t)$ between the actual state and its estimate is shown in Figure 1. The bold line and the dotted line show the errors of e_1 and e_2, respectively. It follows from this that our observer clearly estimates the actual values of the states $x_1,\ x_2$ since the errors e_1 and e_2 converge to zero.

V. CONCLUSIONS

Observer design methods of general nonlinear systems described by T-S fuzzy systems has been proposed. A new non-PDO makes it possible to estimate the state variables. A new multiple Lyapunov matrix approach was introduced to show new observer design conditions, which were shown to be less conservative. Using a numerical example, we showed our observer estimates the states.

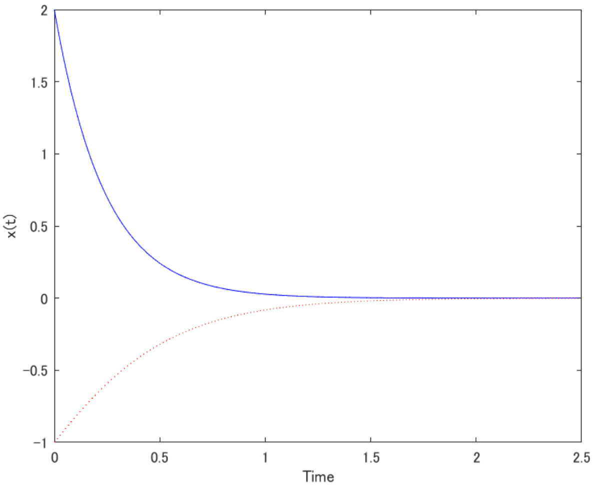

Fig. 1. The estimation errors

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